

A covariant path amplitude description of flavour oscillations: the Gribov–Pontecorvo phase for neutrino vacuum propagation is right

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Abstract. An extended study is performed of geometrical and kinematical assumptions used in calculations of the neutrino oscillation phase. The almost universally employed “equal velocity” assumption, in which all neutrino mass eigenstates are produced at the same time, is shown to underestimate, by a factor of two, the neutrino propagation contribution to the phase. Taking properly into account, in a covariant path amplitude calculation, the incoherent nature of neutrino production as predicted by the standard model, results in an important source propagator contribution to the phase. It is argued that the commonly discussed Gaussian “wave packets” have no basis within quantum mechanics and are the result of a confused amalgam of quantum and classical wave concepts.

1 Introduction

The first published calculation of the phase of neutrino oscillations, on the assumption that neutrinos are massive, but ultra-relativistic, was made in 1969 by Gribov and Pontecorvo [1]. No details of the method used to obtain the phase were given. Later publications used this result for phenomenology. In 1976 a paper was published [2] in which an oscillation phase a factor of two smaller than the Gribov–Pontecorvo result was obtained. No comment was made, then or later, on this discrepancy, which remained unnoticed before a paper written recently by the present author [3] in which more bibliographic details of this “missing controversy” can be found. Since the first derivation of the “standard” oscillation phase, as the result of [2] will be referred to in this paper, of the order of a hundred papers have been published devoted to the quantum mechanics (QM) of neutrino oscillations.

The present paper has three aims. The first one is to carefully review the kinematical and geometrical assumptions of previous calculations, particularly with regard to their consistency in terms of an expansion in powers of the neutrino masses. The second is to present a calculation of the oscillation phase, for sources at rest, using a covariant Feynman path amplitude formulation [4, 5] of QM¹. The third is to critically discuss treatments, in the literature, of the QM of neutrino oscillations, in the light of both the kinematical and geometrical findings of the present paper

and the physical insights provided by the path amplitude treatment.

The calculations presented here are only a subset of those to be found in a previous paper by the present author [3], results of which are also summarised in a short letter [6]. The calculations in [3] include neutrino oscillations following decay at rest of pions, muons and β unstable nuclei, muon oscillations following pion decay at rest, and neutrino and muon oscillations following decay in flight of ultra-relativistic pions. Also discussed in [3] are different mechanisms (both coherent and incoherent) that may contribute to damping of the oscillations. An attempt has been made, in the present paper, to give a more pedagogical presentation in which the crucial underlying physical assumptions used are spelt out as clearly as possible.

Reference [3] contains already a extensive critical review of the previous literature. In the present paper, the further discussion has been largely motivated and orientated by the reports of several anonymous reviewers of [3, 6]. Since writing these papers I became aware of the important work of Shrock [7, 8] who pointed out, more

¹ Feynman’s original work on this subject cited in [4, 5] concerned only non-relativistic QM, but the fundamental formula (67), on which the work presented in the present paper is based, holds also in relativistic QM. The approach should be more properly termed a “path amplitude” rather than a “path integral” one as no attempt is made to evaluate the latter in closed mathematical form, as is done, for example, in the discussion of bound state problems

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Table 1. Values of the flavour-mass mixing amplitude $\langle p|q \rangle$

p	1	2	e	μ
q				
1	0	0	$\cos \theta$	$-\sin \theta$
2	0	0	$\sin \theta$	$\cos \theta$
e	$\cos \theta$	$\sin \theta$	0	0
μ	$-\sin \theta$	$\cos \theta$	0	0

than twenty years ago, the incoherent² nature of the weak decay processes in which neutrino mass eigenstates are produced. This was indeed one of the crucial assumptions for the calculations presented in [3, 6]. I also now fully understand the (simple) reason why the standard oscillation phase has been obtained in essentially all published derivations since [2]. Indeed, as will be demonstrated, the wrong (standard) instead of the correct (Gribov–Pontecorvo) neutrino propagation phase has been universally obtained due to the universal failure to use the correct description of the weak neutrino production processes as formulated by Shrock [7, 8].

For simplicity, only two-flavour mixing will be considered, so that the mixing amplitudes: $\langle i|\alpha \rangle$ (where $i (= 1, 2)$ refers to neutrino mass eigenstates ν_1, ν_2 and $\alpha (= e, \mu)$ refers to the flavour of the charged lepton participating in the weak interaction process) may be taken to be real. Unitarity then enables all mixing amplitudes to be described in terms of a single angular parameter, θ , as shown in Table 1.

Note that there are no “neutrino flavour eigenstates”. The amplitude $\langle 1|\mu \rangle$, for example, is the amplitude to produce the neutrino mass eigenstate ν_1 in association with (or in the decay of) a muon. Thus the flavour-mass mixing amplitudes fix the strength of the charged current couplings to the neutrino mass eigenstates [7, 8]. For example, in the decays $\pi^+, K^+, B^+ \rightarrow \mu^+ \nu_i$ the leptonic charged current is $\bar{\nu}_i \gamma_\rho (1 - \gamma_5) U_{\mu i} \mu$ ($i = 1, 2$) to be compared with the quark currents: $\bar{q} \gamma_\rho (1 - \gamma_5) V_{uq} u$ ($q = d, s, b$), where $U_{\mu i} = U_{i\mu} \equiv \langle 1|\mu \rangle$ is an element of the Maki–Nakagawa–Sakata (MNS) [9] lepton flavour/mass mixing matrix, and V_{uq} is an element of the Cabibbo–Kobayashi–Maskawa (CKM) [10] flavour/mass mixing matrix in the quark sector. The couplings of antineutrinos to charged antileptons are the same as those of neutrinos to charged leptons in the two-flavour mixing case.

The amplitude to produce a charged lepton of flavour β by the interaction of a neutrino mass eigenstate i produced in a decay involving a charged lepton³ of flavour α may,

in general, be written as

$$A_i(\beta \leftarrow \alpha) = A_0 \exp(-i\Delta\phi_i) \langle \beta|i \rangle \langle i|\alpha \rangle. \quad (1)$$

The probability to observe a charged lepton of flavour β , in the case that the neutrino mass eigenstates are not distinguished, is then, by superposition,

$$\begin{aligned} P_{\beta\alpha} &= |A_1(\beta \leftarrow \alpha) + A_2(\beta \leftarrow \alpha)|^2, \\ &= |A_0|^2 [(\langle \beta|1 \rangle \langle 1|\alpha \rangle)^2 + (\langle \beta|2 \rangle \langle 2|\alpha \rangle)^2 \\ &\quad + 2\langle \beta|1 \rangle \langle \beta|2 \rangle \langle 1|\alpha \rangle \langle 2|\alpha \rangle \cos(\Delta\phi_1 - \Delta\phi_2)]. \end{aligned} \quad (2)$$

The “neutrino oscillation phase” ϕ_{12} is defined as

$$\phi_{12} \equiv \Delta\phi_1 - \Delta\phi_2. \quad (3)$$

The Gribov–Pomeranchuk value of the oscillation phase is

$$\phi_{12}^{\text{GP}} = \frac{\Delta m^2 L}{p_\nu}, \quad (4)$$

whereas the standard result is

$$\phi_{12}^{\text{stand}} = \frac{\Delta m^2 L}{2p_\nu}. \quad (5)$$

In (4) and (5), $\Delta m^2 \equiv m_1^2 - m_2^2$ where m_1 and m_2 are the neutrino masses, L is the source–detector distance and p_ν is the measured neutrino momentum. Units with $\hbar = c = 1$ are used throughout this paper. The word “neutrino”, without further qualification, stands for “neutrino mass eigenstate”. The plan of this paper is as follows: In Sects. 2, 3 and 4 various kinematical and geometrical approximations that have been used in neutrino oscillation calculations are discussed. In particular it will be examined whether particular approximations retain, or not, all the leading order $O(m^2)$ terms, or contribute only negligible $O(m^4)$ corrections. The Lorentz invariant nature of the oscillation phase is fully exploited in these purely mathematical considerations. Section 5 describes the calculation of the oscillation phase using the covariant Feynman path amplitude method. Section 6 is devoted to a discussion of previous treatments of the QM of neutrino oscillations. In particular, the physical basis of widely used Gaussian wave packet models is questioned. Section 7 contains a brief summary and outlook.

2 Lorentz invariant plane wave propagation

In this case, the one-dimensional propagation amplitude for a particle of mass m_i over space and time intervals, Δx_i and Δt_i , is given by

$$\begin{aligned} P(\Delta x_i, \Delta t_i, m_i) &= P_0 \exp \{-i[E_i \Delta t_i - p_i \Delta x_i]\} \\ &= P_0 \exp \{-im_i \Delta \tau_i\} \\ &\equiv P_0 \exp \{-i\Delta\phi_i\}. \end{aligned} \quad (6)$$

² The word “incoherent” here means that the different mass eigenstates are produced in independent physical processes, not in a coherent “flavour eigenstate” that is a superposition of the mass eigenstates. A “coherent source” is defined as one in which the amplitudes corresponding to different decay times have a well-defined phase difference

³ In the decay process, this charged lepton, together with the neutrino ν_i , form a charged current that couples to a real or virtual W boson

Here $\Delta\tau_i$ is the proper time interval, in the rest frame of the particle, i , corresponding to space and time intervals Δx_i and Δt_i in the laboratory frame, so that

$$\Delta\tau_i^2 = (\Delta t_i)^2 - (\Delta x_i)^2. \quad (7)$$

The relativistic relation between energy, momentum and mass,

$$E_i(p_i, m_i) = \sqrt{p_i^2 + m_i^2}, \quad (8)$$

implies a group velocity v_i^G for the travelling wave represented by (6) given by

$$v_i^G \equiv \frac{dE_i}{dp_i} = \frac{p_i}{E_i}. \quad (9)$$

In the following, it is assumed that the neutrino mass eigenstates whose propagation is described by (6), have velocity $v_i = v_i^G$. As correctly emphasised in [11] particle oscillation experiments actually measure the spatial, not the temporal, dependence of quantum mechanical interference effects. Therefore, throughout the kinematical discussions in the present and following sections, a fixed distance, $\Delta x_i = L$, is assumed between the source particle (at rest) and the detection event. With the additional assumption that the propagating neutrino mass eigenstates are on-shell particles, several different exact expressions may be written for the Lorentz invariant phase increment

$$\Delta\phi_i = m_i \Delta\tau_i = \frac{m_i \Delta t_i}{\gamma_i} = \frac{m_i^2 \Delta t_i}{E_i} = \frac{m_i^2 L}{p_i}, \quad (10)$$

where, in the last member of (10), the relation $\Delta t_i = L/v_i$ has been used, and $\gamma = 1/\sqrt{1-v^2} = E/m$ is the usual parameter of special relativity.

The value of the oscillation phase $\phi_{12} \equiv \Delta\phi_1 - \Delta\phi_2$ obtained using different kinematical approximations will now be considered. For this it is useful to introduce the quantity p_0 which is the limiting value of p_i or E_i as $m_i \rightarrow 0$. The following expressions for p_i and E_i may then be written:

$$p_i = p_0 \left\{ 1 - \left(\frac{m_i}{m_S} \right)^2 \left[\frac{1 + R_m^2}{(1 - R_m^2)^2} \right] \right\} + O(m^4), \quad (11)$$

$$E_i = p_0 \left\{ 1 + \left(\frac{m_i}{m_S} \right)^2 \left[\frac{1}{1 - R_m^2} \right] \right\}, \quad (12)$$

where

$$p_0 = \frac{m_S}{2} (1 - R_m^2). \quad (13)$$

Here m_S is the mass of the decaying source particle and $R_m \equiv m_R/m_S$ where m_R is the mass of the particle (or system of particles) recoiling against the neutrino in the decay process. It is also convenient to introduce the average kinematical quantities for the neutrino mass eigenstates:

$$\bar{m} = \frac{(m_1 + m_2)}{2}, \quad (14)$$

$$\bar{p} = \frac{(p_1 + p_2)}{2}, \quad (15)$$

$$\bar{E} = \sqrt{\bar{p}^2 + \bar{m}^2}. \quad (16)$$

If, as is usually done in the literature, common values of both Δx and Δt are assumed for both mass eigenstates, then

$$v_1 = v_2 = \bar{v} = \frac{\Delta x}{\Delta t} = \frac{\bar{p}}{\bar{E}} \quad (17)$$

and

$$\gamma_1 = \gamma_2 = \bar{\gamma} = \frac{\Delta t}{\Delta\tau} = \frac{\bar{E}}{\bar{m}}. \quad (18)$$

It must be remarked at once that such an “equal velocity” hypothesis is in contradiction with the fundamental definitions in (6)–(9). For logical consistency, if Δx and Δt are assumed to be equal for the two mass eigenstates, then also E_i and p_i in (6) should be replaced by \bar{E} and \bar{p} . Then, evidently, no neutrino oscillations would be possible. In fact the equal velocity hypothesis assumes not only (17) but also that the energy and momentum appearing in the 4-vector product in (6) are in accordance with energy-momentum conservation in the decay process. Since $v_i^G = p_i/E_i \neq \bar{v}$, there is then an internal contradiction. Also, because of (17), the corresponding space and time intervals do not respect the equations $v_i^G = \Delta x_i/\Delta t_i$, i.e. the neutrinos do not move along classical trajectories specified by the group velocities v_i^G . Thus the equal velocity hypothesis respects energy-momentum conservation, but not the constraints of space-time geometry, and contains an internal contradiction. The formula (6) combines the quantities $(\Delta x_i, \Delta t_i)$ from space-time geometry and (E_i, p_i) from particle kinematics. The equal velocity hypothesis then treats, in an inconsistent manner, these two sets of quantities. Introducing the notation $\Delta\phi_i(\bar{v})$ for the approximate phase increment calculated with fixed velocity \bar{v} , according to (14)–(18),

$$\Delta\phi_i(\bar{v}) = m_i \Delta\tau = \frac{m_i \Delta t}{\bar{\gamma}} = \frac{m_i^2 \Delta t}{\bar{E}} = \frac{m_i^2 L}{\bar{p}}. \quad (19)$$

Now denoting by the superscripts I, II, III and IV the values of $\phi_{12}(\bar{v})$ calculated using the successive members of (19) the following results are obtained:

$$\begin{aligned} \phi_{12}^I(\bar{v}) &= (m_1 - m_2) \Delta\tau \\ &= (m_1 - m_2) \frac{\bar{m} L}{\bar{p}} = \frac{(m_1^2 - m_2^2) L}{2\bar{p}}, \end{aligned} \quad (20)$$

$$\begin{aligned} \phi_{12}^{II}(\bar{v}) &= (m_1 - m_2) \frac{\Delta t}{\bar{\gamma}} \\ &= (m_1 - m_2) \frac{\bar{m} L}{\bar{p}} = \frac{(m_1^2 - m_2^2) L}{2\bar{p}}, \end{aligned} \quad (21)$$

$$\phi_{12}^{III}(\bar{v}) = (m_1^2 - m_2^2) \frac{\Delta t}{\bar{E}} = \frac{(m_1^2 - m_2^2) L}{\bar{p}}, \quad (22)$$

$$\phi_{12}^{IV}(\bar{v}) = \frac{(m_1^2 - m_2^2) L}{\bar{p}}. \quad (23)$$

The phases ϕ_{12}^{III} and ϕ_{12}^{IV} are found to be equal and a factor of two larger than either ϕ_{12}^{I} or ϕ_{12}^{II} . The different results obtained indicate that some inconsistent approximations must have been made in the passage from the exact expressions for the phase increment in (10) to the approximate ones in (19). To see just where the inconsistency has occurred, and which results, I and II or III and IV, are correct, the calculation is now done using the exact formulae of (10) and, for ease of comparison with the approximate expressions in (19) the neutrino masses are written as

$$m_i = \bar{m} + \delta_i, \quad (24)$$

where

$$\delta_1 = -\delta_2 = \frac{m_1 - m_2}{2}. \quad (25)$$

Thus δ_i are correction terms, relating the approximate relations in (19) to the exact ones in (10). Denoting by $\phi_{12}(\text{exact})$ the interference phase calculated using (10) then

$$\begin{aligned} \phi_{12}(\text{exact}) &= m_1 \Delta\tau_1 - m_2 \Delta\tau_2 \\ &= m_1 \frac{\Delta t_1}{\gamma_1} - m_2 \frac{\Delta t_2}{\gamma_2} = \left(\frac{m_1^2}{p_1} - \frac{m_2^2}{p_2} \right) L \\ &= \frac{(m_1^2 - m_2^2)}{\bar{p}} L + O(m^4). \end{aligned} \quad (26)$$

Thus the result obtained is in agreement, at $O(m^2)$, with the approximate calculations in cases III and IV. Using now (24) and (25)

$$\begin{aligned} \phi_{12}(\text{exact}) &= \left[\frac{(\bar{m} + \delta_1)^2}{p_1} - \frac{(\bar{m} + \delta_2)^2}{p_2} \right] L \\ &= \left[\frac{(\bar{m} + \delta_1)^2 - (\bar{m} + \delta_2)^2}{\bar{p}} \right] L + O(m^4) \\ &= \frac{(m_1^2 - m_2^2)}{2\bar{p}} L + \frac{(m_1 - m_2)\bar{m}}{\bar{p}} L + O(m^4). \end{aligned} \quad (27)$$

Comparing the second term on the RHS this equation with (20) or (21) it can be seen that the correction terms in (24) have the effect of doubling the approximate results $\phi_{12}^{\text{I}}(\bar{v})$ and $\phi_{12}^{\text{II}}(\bar{v})$. Thus important $O(m^2)$ contributions are neglected in (20) and (21). The reason why the constant velocity approximation is so poor in (20) and (21) becomes evident on inspection of (18). Unlike in the case of neutral kaons or B^0 mesons, the neutrino masses may be widely different, so that although, for highly relativistic neutrinos, the velocities of the mass eigenstates may be very similar, this is not the case for the Lorentz- γ factors in (18) that are inversely proportional to the neutrino masses. The important difference between the exact and approximate calculations occurs in the very first member of (20). Indeed, as is shown by (26) and (20):

$$\begin{aligned} \phi_{12}(\text{exact}) - \phi_{12}^{\text{I}}(\bar{v}) &= m_1 \Delta\tau_1 - m_2 \Delta\tau_2 - (m_1 - m_2) \Delta\tau \\ &= \frac{(m_1^2 - m_2^2)}{2\bar{p}} L + O(m^4). \end{aligned} \quad (28)$$

In fact, in (22) and (23), the approximate Lorentz- γ factor $\bar{\gamma} = \bar{E}/\bar{m}$ is replaced by the still approximate but mass dependent γ factors \bar{E}/m_i which differ from the exact ones E_i/m_i by corrections of only $O(m^2)$. Thus, up to corrections of $O(m^4)$, the same result is obtained as the exact formula. On the other hand the difference between $1/\bar{\gamma}$, used in (21), and $1/\gamma_1$ or $1/\gamma_2$ used in (10), is of $O(m)$ and so cannot be neglected in an $O(m^2)$ calculation of the oscillation phase.

It is instructive to calculate the oscillation phase directly in the laboratory system, using (6), rather than by Lorentz transformation from the neutrino center of mass to the laboratory system as in (10) and (19). Considering first the equal velocity case:

$$\begin{aligned} \phi_{12}^{\text{LAB}}(\bar{v}) &= (E_1 - E_2) \Delta t - (p_1 - p_2) L \\ &= \left[(E_1 - E_2) \frac{1}{\bar{v}} - (p_1 - p_2) \right] L. \end{aligned} \quad (29)$$

Using the relations

$$E_i = \sqrt{p_i^2 + m_i^2} = p_i + \frac{m_i^2}{2p_i} + O(m^4), \quad (30)$$

$$\bar{v} = 1 - \frac{\bar{m}^2}{2\bar{p}^2} + O(m^4). \quad (31)$$

and noting that, from (11) and (15) that $p_i = \bar{p} + O(m^2)$, (29) may be written as

$$\begin{aligned} \phi_{12}^{\text{LAB}}(\bar{v}) &= \left(\frac{m_1^2}{2p_1} - \frac{m_2^2}{2p_2} \right) L + O(m^4) \\ &= \frac{(m_1^2 - m_2^2)}{2\bar{p}} L + O(m^4), \end{aligned} \quad (32)$$

in agreement with (20) and (21). The exact formula gives

$$\begin{aligned} \phi_{12}^{\text{LAB}}(\text{exact}) &= (E_1 \Delta t_1 - E_2 \Delta t_2) - (p_1 - p_2) L \\ &= \left[\frac{E_1}{v_1} - p_1 - \frac{E_2}{v_2} + p_2 \right] L \\ &= \left[\frac{m_1^2}{p_1} - \frac{m_2^2}{p_2} \right] L \\ &= \frac{(m_1^2 - m_2^2)}{\bar{p}} L + O(m^4), \end{aligned} \quad (33)$$

in agreement with (22) and (23). It can be seen that

$$\begin{aligned} \phi_{12}^{\text{LAB}}(\text{exact}) - \phi_{12}^{\text{LAB}}(\bar{v}) &= (E_1 \Delta t_1 - E_2 \Delta t_2) - (E_1 - E_2) \Delta t \\ &= \frac{(m_1^2 - m_2^2)}{2\bar{p}} L + O(m^4). \end{aligned} \quad (34)$$

Again, the ‘‘equal velocity’’ formula is seen to neglect $O(m^2)$ contributions that double the value of the oscillation phase.

The neglect of important $O(m^2)$ terms when the equal velocity hypothesis is made especially transparent on re-writing (26) in the following way:

$$\begin{aligned}\phi_{12}(\text{exact}) &= m_1 \Delta\tau_1 - m_2 \Delta\tau_2 \\ &= (m_1 - m_2) \Delta\tau_1 + m_2 (\Delta\tau_1 - \Delta\tau_2).\end{aligned}\quad (35)$$

Since

$$\Delta\tau_i = \frac{m_i L}{p_i} = \frac{m_i L}{p_0} + O(m^3), \quad (36)$$

(35) can be written as

$$\phi_{12}(\text{exact}) = \frac{(m_1 - m_2)}{p_0} [m_1 + m_2] L + O(m^4). \quad (37)$$

The equal velocity hypothesis now implies

- (a) Neglect of $\Delta\tau_1 - \Delta\tau_2$ i.e. of the second term in the square brackets in (37);
- (b) making the replacement $m_1/p_0 \rightarrow \bar{m}/\bar{p}$ in the remaining term of (37).

It is now obvious that (a) and (b) neglect (different) $O(m^2)$ terms; a term $\simeq m_2(m_1 - m_2)$ for (a) and a term $\simeq (m_1 - \bar{m})(m_1 - m_2) = (m_1 - m_2)^2/2$ for (b). The sum of these neglected terms is $\simeq (m_1^2 - m_2^2)/2 = \bar{m}(m_1 - m_2)$, equal to the terms retained by the equal velocity hypothesis.

Thus the essentially universally employed “equal velocity” assumption results in the neglect of $O(m^2)$ terms in the calculation of the vacuum oscillation phase. The existence of these terms explains the factor of two difference between the original calculation of the oscillation phase by Gribov and Pontecorvo (which is the correct $O(m^2)$ result) and the “standard phase” given by all calculations that make the “equal velocity” assumption.

The calculation of the oscillation phase is now repeated assuming, instead of equal velocities for the neutrino mass eigenstates, either equal momenta or equal energies. For equal momenta the following relations hold:

$$L = \Delta t_i(p) v_i(p), \quad (38)$$

$$E_i(p) = \sqrt{p^2 + m_i^2}, \quad (39)$$

$$v_i(p) = \frac{p}{E_i(p)}, \quad (40)$$

$$\gamma_i(p) = \frac{E_i(p)}{m_i}. \quad (41)$$

These yield, for the invariant phase increment,

$$\Delta\phi_i(p) = m_i \Delta\tau_i(p) = \frac{m_i \Delta t_i(p)}{\gamma_i(p)} = \frac{m_i^2 \Delta t_i(p)}{E_i(p)} = \frac{m_i^2 L}{p}. \quad (42)$$

Comparing with (10) it can be seen that each member of the two equations differ only by terms of $O(m^4)$. Thus, at $O(m^2)$ the Gribov–Pontecorvo result (26) is obtained for the oscillation phase. For the equal energy case

$$L = \Delta t_i(E) v_i(E), \quad (43)$$

$$p_i(E) = \sqrt{E^2 - m_i^2}, \quad (44)$$

$$v_i(E) = \frac{p_i(E)}{E}, \quad (45)$$

$$\gamma_i(E) = \frac{E}{m_i}, \quad (46)$$

$$\begin{aligned}\Delta\phi_i(E) &= m_i \Delta\tau_i(E) = \frac{m_i \Delta t_i(E)}{\gamma_i(E)} = \frac{m_i^2 \Delta t_i(E)}{E} \\ &= \frac{m_i^2 L}{p_i(E)}.\end{aligned}\quad (47)$$

Again, all terms in last equation differ from the corresponding ones in the exact expression (10) only by terms of $O(m^4)$ so that the oscillation phase agrees with the value in (26). It is easily seen that this result is also obtained in the case of either equal momenta or equal energies if the oscillation phase is calculated directly in the laboratory system as in (33).

It may be shown, in a similar way, that once a constant value of Δt is assumed, the standard oscillation phase is obtained independently of whether the energy and momentum are calculated on the assumption of exact energy-momentum conservation, or whether the equal momentum and different energy or different momentum and equal energy hypotheses are used.

Thus, the “standard” oscillation phase is obtained only in the case of the “equal velocity” hypothesis. In all cases: “equal velocity”, “equal momentum” and “equal energy”, an unphysical assumption is being made (i.e. one that violates conservation of energy and momentum). The study just presented shows that in the latter two cases the error of the approximation is only of $O(m^4)$ in the oscillation phase, and so is negligible at $O(m^2)$. However, in the former case, the neglected terms are of $O(m^2)$ and so lead to an incorrect result at this order.

Finally, in this section it will be found of interest for the subsequent discussion of independent temporal or spatial propagation to calculate the separate temporal (T) and spatial (S) parts, in the laboratory system, of the Lorentz invariant oscillation phase ϕ_{12} :

$$\begin{aligned}\phi_{12}^{\text{LAB}}(\text{T}) &= (E_1 \Delta t_1 - E_2 \Delta t_2) \\ &= \left(\frac{E_1^2}{p_1} - \frac{E_2^2}{p_2} \right) L \\ &= \left(p_1 - p_2 + \frac{m_1^2}{p_1} - \frac{m_2^2}{p_2} \right) L\end{aligned}\quad (48)$$

$$= \left[1 - \frac{(1 + R_m^2)}{4} \right] \frac{(m_1^2 - m_2^2)}{p_0} L + O(m^4),$$

$$\begin{aligned}\phi_{12}^{\text{LAB}}(\text{S}) &= -(p_1 - p_2) L \\ &= \frac{(1 + R_m^2)}{4p_0} (m_1^2 - m_2^2) L + O(m^4).\end{aligned}\quad (49)$$

On adding (48) to (49) it can be seen that the Lorentz invariant oscillation phase originates entirely from the tem-

poral part, the spatial part being exactly cancelled by a term in the temporal one.

3 Gaussian wave packet propagation

Very often in the literature, following [12], the plane wave propagator of (6) has been modified by the introduction of a multiplicative Gaussian spatial wave packet, $G(\Delta x, \Delta t, v_i, \sigma_x)$:

$$P_{\text{WP}}(\Delta x, \Delta t, m_i, v_i, \sigma_x) = G(\Delta x, \Delta t, v_i, \sigma_x) P(\Delta x, \Delta t, m_i), \quad (50)$$

where $P(\Delta x, \Delta t, m_i)$ is given in (6) and

$$G(\Delta x, \Delta t, v_i, \sigma_x) = (\sqrt{2\pi}\sigma_x)^{-\frac{1}{2}} \exp \left[- \left(\frac{\Delta x - v_i \Delta t}{2\sigma_x} \right)^2 \right]. \quad (51)$$

In this formulation the intervals Δx and Δt are assumed to be the same for all mass eigenstates, so that the equal velocity hypothesis is tacitly introduced. To calculate the oscillation phase, ϕ_{12}^{WP} an integration over the transit time in the range $-\infty < \Delta t < \infty$ is performed, leading to the result [12]

$$\phi_{12}^{\text{WP}} = \left[\frac{E_1 - E_2}{\tilde{v}} - (p_1 - p_2) \right] L, \quad (52)$$

where

$$\tilde{v} = \frac{v_1^2 + v_2^2}{v_1 + v_2} = 1 - \frac{1}{2} \left(\frac{m_1^2}{p_1^2} + \frac{m_2^2}{p_2^2} \right) + O(m^4). \quad (53)$$

It may be noted that the effective average velocity \tilde{v} is *lower* than that of either mass eigenstate. This is presumably due to the contribution of the unphysical⁴ negative region in the integration over Δt . Using the relation

$$E_i = p_i + \frac{m_i^2}{2p_i} + O(m^4). \quad (54)$$

Equation (52) may be written

$$\begin{aligned} \phi_{12}^{\text{WP}} &= \left\{ \left[(p_1 - p_2) + \frac{1}{2} \left(\frac{m_1^2}{p_1} - \frac{m_2^2}{p_2} \right) \right] \left[1 + \frac{1}{2} \left(\frac{m_1^2}{p_1^2} + \frac{m_2^2}{p_2^2} \right) \right] \right. \\ &\quad \left. - (p_1 - p_2) \right\} + O(m^4) \\ &= \left[\frac{1}{2} \left(\frac{m_1^2}{p_1} - \frac{m_2^2}{p_2} \right) + \frac{(p_1 - p_2)}{2} \left(\frac{m_1^2}{p_1^2} + \frac{m_2^2}{p_2^2} \right) \right] L \\ &\quad + O(m^4) \\ &= \frac{(m_1^2 - m_2^2)}{2\bar{p}} L + O(m^4), \end{aligned} \quad (55)$$

⁴ Note that the Gaussian variation of Δt , for fixed source-detector distance $\Delta t = L$, assumed in this calculation, is at variance with the known space-time structure of the decay and detection events, i.e. an exponential distribution of decay times followed by a mass dependent propagation time Δt_i

where (11) has been used to write p_i in terms of p_0 and m_i . The “wave-packeted” propagator thus gives the same oscillation phase at $O(m^2)$ as the unmodified invariant plane wave propagator when the same equal velocity hypothesis is made. In both cases important $O(m^2)$ terms are neglected in comparison with the exact calculation.

In a more elaborate recent calculation [13] where wave packets are associated not only with the neutrinos, but also with all particles participating in the production and decay processes, a formula for the phase increment identical to (50) was obtained except that the width of the spatial wave packet σ_x is replaced by a parameter η that depends on the widths of the wave packets of all participating particles. Since the oscillation phase (52) does not depend on this parameter the same “standard” result is obtained as in [12]. The results obtained with different mean velocities, \bar{v} given by (14)–(17) and \tilde{v} given by (53) are the same. Indeed, it is clear from the derivation of (55) from (52) that a mean velocity $\langle v \rangle$ defined by any formula of the type

$$\langle v \rangle = 1 - \frac{\alpha m_1^2 + \beta m_1 m_2 + \gamma m_2^2}{\bar{p}^2} + O(m^4), \quad (56)$$

where α , β and γ are arbitrary coefficients of order unity, will yield the result (55). However, in the derivation of the same formula in (20) and (21) this result requires that $\bar{m} = (m_1 + m_2)/2$. If instead the value $\tilde{m} = \sqrt{m_1^2 + m_2^2}$, suggested by (53), is used, then a different result is obtained:

$$\begin{aligned} \phi_{12}^{\text{I}}(\tilde{v}) &= \phi_{12}^{\text{II}}(\tilde{v}) \\ &= \frac{(m_1 - m_2)(\sqrt{m_1^2 + m_2^2})L}{\bar{p}} + O(m^4). \end{aligned} \quad (57)$$

Thus, unlike in the case of the invariant plane wave, the wave-packet treatment gives inconsistent results for the oscillation phase in the neutrino centre of mass and laboratory systems.

4 Independent temporal or spatial propagation

Many descriptions of neutrino oscillations (especially concise presentations in review articles concerned mainly with the description of experimental results, or theoretical models of neutrino masses) do not use a Lorentz invariant description but consider instead independent temporal (TE) or spatial (SE) evolution of the wavefunctions of the neutrino mass eigenstates. It is in this way that the standard formula for the oscillation phase of (5) was first derived in [2]. A simple Schrödinger time evolution is used giving the temporal propagator

$$P_{\text{TE}}(\Delta t, m_i) = P_0 \exp[-iE_i \Delta t], \quad (58)$$

or

$$\Delta\phi_i(\text{TE}) = E_i \Delta t. \quad (59)$$

To relate the time interval Δt (assumed to be the same for both neutrino mass eigenstates) to the experimentally

measured source–detector separation L , it is further assumed that the neutrinos are ultra-relativistic and so move at the speed of light, i.e. $\Delta t = L$. To derive the oscillation phase the eigenstates are then assumed to have equal momentum, p_ν , but different energies. As first pointed out in [14] this assumption is in contradiction with energy-momentum conservation in the decay process. This is easily seen by inspection of (11) and (12) above. Indeed, the violation of energy-momentum conservation is an $O(m^2)$ effect. Still, making this assumption,

$$E_i = p_\nu + \frac{m_i^2}{2p_\nu} + O(m^4), \quad (60)$$

so that (59) gives

$$\begin{aligned} \phi_{12}(TE) &= \Delta\phi_1(TE) - \Delta\phi_2(TE) \\ &= \frac{(m_1^2 - m_2^2)L}{2p_\nu} + O(m^4). \end{aligned} \quad (61)$$

This calculation is wrong, firstly because it does not respect Lorentz invariance and secondly because the kinematical approximations made (neutrinos of equal momenta moving at the speed of light) neglect important $O(m^2)$ terms, as can be seen by comparing the exact expression of the temporal part $\phi_{12}^{\text{LAB}}(T)$ in the laboratory system of the Lorentz invariant oscillation phase in (48) with (61).

Returning now to the question of the compatibility of (58) with special relativity, such a formula is a reasonably good approximation to the exact, Lorentz invariant, result in the non-relativistic (NR) limit where $p_i \ll m_i$. In this case

$$\Delta\phi_i^{\text{LAB}}(\text{NR}) = E_i \Delta t_i = \frac{m_i^2 L}{p_i} \left(1 + \frac{p_i^2}{m_i^2} \right), \quad (62)$$

so that, up to a fractional correction p_i^2/m_i^2 , the non-relativistic result is the same as the exact one (10), and the same result, (26), is obtained for ϕ_{12} . The different value obtained in (61) evidently is a consequence of the “equal time” assumption $\Delta t_1 = \Delta t_2 = \Delta t$, which results in the neglect of important $O(m^2)$ terms. It should be stressed that the use of (61) or (62) to describe ultra-relativistic neutrinos is wrong in just the same way that the use of the non-relativistic formula for the total energy of a particle, $E = m + T$, where $T = mv^2/2$, is wrong when $v \simeq 1$.

Other authors [11, 15, 16] motivated by the correct observation that flavour oscillation experiments actually measure quantum interference effects as a function of space, not time, have proposed to describe neutrino oscillations in terms of only a spatial evolution of the wavefunction:

$$P_{\text{SE}} = P_0 \exp[ip_i L], \quad (63)$$

or

$$\Delta\phi_i(\text{SE}) = -p_i L. \quad (64)$$

Now, making the additional assumption (incompatible with energy-momentum conservation in the decay process) of equal energies, E_ν , so that

$$p_i = E_\nu - \frac{m_i^2}{2E_\nu} + O(m^4), \quad (65)$$

(64) gives

$$\begin{aligned} \phi_{12}(\text{SE}) &= \Delta\phi_1(\text{SE}) - \Delta\phi_2(\text{SE}) \\ &= \frac{(m_1^2 - m_2^2)L}{2E_\nu} + O(m^4) \\ &= \frac{(m_1^2 - m_2^2)L}{2p_\nu} + O(m^4), \end{aligned} \quad (66)$$

in agreement with (61). As in the case of independent time evolution, (64) is wrong, firstly, because it does not respect Lorentz invariance, and secondly kinematical assumptions are made (equal energies and different momenta) leading to a result differing by $O(m^2)$ terms from the spatial part of the exact invariant result, as can be seen by comparing (66) and (49). Unlike for the case of time evolution, there is no kinematical domain in which an equation similar to (64) is a good approximation. It is tantamount to performing a kinematical calculation with 4-vectors, the temporal components of which have all been arbitrarily set to zero. In all cases it yields a very bad approximation. Actually, to derive (63) [11, 15, 16] the Lorentz invariant plane wave was written down and equal velocities, i.e. $\Delta t_1 = \Delta t_2 = \Delta t$, as well as equal energies were assumed for both mass eigenstates so that the temporal contributions cancel exactly in the interference phase. This is equivalent to assuming the purely spatial propagation of (63). The physical arguments given in [11, 15, 16] to justify the equal energy hypothesis are further critically examined in Sect. 6 below.

It should be noted that both the TE and SE hypotheses contain a further internal contradiction, different from that of the equal velocity hypothesis discussed in Sects. 3 and 4 above. In the latter, the correct values of E_i and p_i , as given by energy-momentum conservation, are assigned in the plane wave propagator; only the geometrically inconsistent values $\Delta t_1 = \Delta t_2 = \Delta t$ are assumed. For the TE and SE hypotheses, not only are the values of Δt_i inconsistent with the relation $\Delta t_i = L/v_i = LE_i/p_i$, but energy-momentum conservation is also violated by the “equal momentum” or “equal energy” hypotheses. The latter assumptions, however, only generate shifts of $O(m^4)$ in the oscillation phase as compared to the result obtained assuming exact energy-momentum conservation.

5 The quantum mechanics of flavour oscillations

In this section, a general discussion of the underlying principles of flavour oscillation calculations in the Feynman path amplitude formulation of QM is first given before deriving the oscillation phase for a simple two-flavour problem with the source particle at rest. More refined calculations performed in a similar manner may be found in [3].

After the pioneering conceptual work of Planck, Einstein and Bohr, QM was formulated in several independent ways; most importantly, this was done by Heisenberg, Schrödinger and Feynman. The Heisenberg (matrix mechanics) and Schrödinger (wave mechanics) formulations

originally competed for the description of atomic physics. However, the greater similarity with classical physics, the less abstract nature and superior calculational power of the wave mechanics approach resulted in this soon being adopted as the standard one, both in atomic physics research and in text books on non-relativistic QM. Wave mechanics is particularly well adapted to subjects such as atomic, molecular and nuclear physics, where the solution of bound state problems plays an important role; a domain that may be called “quantum statics”. In the present writer’s opinion it is, however, less well suited to “quantum dynamics”, where predictions concerning measurements of events at different space-time points are required. Flavour oscillation experiments are of just this latter type. The formulation of QM most naturally adapted to such problems is the Feynman path integral.

The story of this approach is really that of the re-introduction of space-time into microscopic physics after the quantum revolution of the early 20th Century. Bohr declared in his “Copenhagen interpretation” talk at Lake Como [17] that space and time were outmoded classical categories in the quantum world⁵. The first important step in the direction of the rehabilitation of space-time was Dirac’s seminal paper on the Lagrangian in QM [18], that was the basis Feynman’s formulation of the principles of QM specifically in terms of probability amplitudes corresponding to particles moving in space-time [4,5]. The essence of this approach is contained in a single formula, the importance of which, for a fundamental understanding of QM, had already been stressed by Heisenberg in 1929 [19]:

$$P_{FI} = \sum_m \sum_l \left| \sum_{k_n} \dots \sum_{k_2} \sum_{k_1} \langle f_m | k_n \rangle \dots \langle k_2 | k_1 \rangle \langle k_1 | i_l \rangle \right|^2. \quad (67)$$

This equation gives the probability of transitions between a group $I = \sum_l i_l$ of initial states and a group $F = \sum_m f_m$ of final states. The use of this formula to describe a quantum experiment requires knowledge of both the initial (prepared) and final (measured) states. Indeed, the corresponding quantum experiment is defined by the experimenter’s choice of these states. Only when they are specified is a meaningful comparison of theory with experiment possible. Since typical experimental conditions do not permit one to either prepare or measure a single quantum state, the (incoherent) sums over l and m in (67) are necessary. These sums are defined by purely experimental criteria (detector sizes or resolution) and are unrelated to microscopic, quantum level, parameters. The

states $|k_n\rangle \dots |k_2\rangle, |k_1\rangle$ (of which, in the detailed description of any actual space-time quantum experiment, there are, actually, a multiple infinity) represent *unobserved* intermediate states. In the case of particle trajectories in space-time, described by space-time coordinates, x_i , of interest for the present discussion, the phase to be assigned to any given path amplitude, say, $\langle f_m | x_n \rangle \dots \langle x_2 | x_1 \rangle \langle x_1 | i_l \rangle$ is given by the corresponding classical action divided by \hbar . That is, the integral of the classical Lagrangian (divided by \hbar) along the classical particle trajectory corresponding to the sequence of space-time points x_1, x_2, \dots, x_n where $t_1 < t_2 < \dots < t_n$. An important feature of (67) is that interference occurs only between amplitudes corresponding to *different unobserved intermediate states*, never between different initial and final states.

For the flavour oscillation problem, the general path integral formula (67) may be simplified considerably. The initial and final states correspond to the space-time positions⁶ of the source particle and the detection event. As the particles propagate over macroscopic distances, in free space, they follow classical (straight line) trajectories. The corresponding Feynman path integral then reduces to a Green function, or propagator, that is the Fourier transform of the well-known momentum space propagator of a free particle in quantum field theory. The explicit, general, expression for the invariant space-time propagator of a fermion was given in one of Feynman’s early papers on QED [20]. More recently, it has been rederived directly, from the covariant path integral, for an arbitrary massive particle, by Mohanty [21]. Apart from solid angle correction factors, that are not relevant to the conventional 1-dimensional discussion of flavour oscillations, the propagator for on-shell particles, or for virtual particles over macroscopic time-like invariant intervals such that $c\Delta\tau \gg 1/m_P$ (where m_P is the pole mass of the particle) is given, for a stable particle⁷, up to constant factors, by the simple expression $\exp(-im_P\Delta\tau)$. This means that it is mathematically identical to the “Lorentz invariant plane wave” of (6). However, its physical meaning: a Green function that gives the amplitude for a particle to be found at a given space-time point, when, at some prior time, the particle was situated at another, well-defined, spatial position, is quite different to that of the “plane waves” (energy-momentum eigenstates) of conventional Schrödinger–Born wave mechanics. The latter, interpreted as wavefunctions according to the Born prescription, are not square integrable, and so are devoid of any information concerning the position of the particle. In the case of flavour oscillation experiments, on the other hand, the

⁵ In [17], Bohr wrote “Notwithstanding the difficulties which, hence, are involved in the formulation of the quantum theory, it seems, as we shall see, that its essence may be expressed in the so-called quantum postulate which attributes to any atomic process an essential discontinuity, or rather individuality, completely foreign to the classical theories and symbolised by Planck’s quantum of action. This postulate implies a renunciation as regards the causal space-time co-ordination of physical processes”

⁶ In the following, quantum states will be specified in “space-time” rather than in “configuration space” as is conventionally done. The latter definition is convenient when describing, say, the spatial wavefunction of a bound state, such as the hydrogen atom, which requires 6 spatial coordinates for its complete specification. However, flavour oscillation experiments relate events at different space-time, not configuration space, positions

⁷ An unstable particle is described by adding a negative imaginary term to the pole mass: $m_P \rightarrow m_P - i\Gamma/2$

experimental knowledge that can be obtained concerning the position of the propagating particles, once they are created by the decay process, is essentially classical, i.e., unaffected by uncertainties of a purely quantum mechanical nature. The meaning of this remark will be clarified by a concrete example to be given below. In fact “quantum uncertainty” as manifested in the Heisenberg uncertainty relations results only in the uncertainty in the time at which the propagating particle is produced by the unstable source particle. The associated spread in the physical mass of the source particle, and of any unstable recoil particles, then produces, via energy-momentum conservation in the decay process, momentum or velocity smearing of the neutrino mass eigenstates.

To take a specific example, consider the decay of a pion at rest: $\pi^+ \rightarrow \mu^+ \nu_i$. As the pion is unstable, it has, in general, a physical mass W_π that is different from the most likely physical mass, the pole mass, m_π . Similarly, the physical mass W_μ is different from the pole mass m_μ . The Breit–Wigner widths, Γ , of the physical mass distributions are related to the particle mean lifetimes, τ , by the energy-time uncertainty relation $\Gamma\tau = 1$. Since the initial state of the path amplitudes, that of the source pion, is the same in all path amplitudes, the kinematical effects of the non-equality of W_π and m_π will be the same for all path amplitudes, i.e. the resultant velocity smearing of the neutrinos is incoherent. On the other hand, since the recoil muon is unobserved, its physical mass may be different in the paths corresponding to different mass eigenstates. The corresponding Breit–Wigner amplitudes will then create a coherent velocity (or momentum) wave packet for each eigenstate. This is a simple application of (67). The spatial trajectory, for fixed momentum, p , yielding the path amplitude

$$\text{PA} = \langle f_m | x_n, p \rangle \dots \langle x_2, p | x_1, p \rangle \langle x_1, p | i_l \rangle, \quad (68)$$

is replaced by the sum of path amplitudes (path integral):

$$\text{PI} = \sum_q \text{PA}_q = \sum_q \langle f_m | x_n, p_q \rangle \dots \langle x_2, p_q | x_1, p_q \rangle \langle x_1, p_q | i_l \rangle. \quad (69)$$

As shown in [3], the effect of such wave packets on the oscillation phenomenon is minute, in particular for the case of pion decay, as well as for neutrino oscillation experiments in general. As discussed below, it may, however, become more important for quark flavour oscillations where the spread in the physical masses of the propagating particles is of the same order as their mass difference.

All the relevant constraints imposed by the Heisenberg uncertainty relations have been taken into account in the above discussion. In particular, as will be considered in more detail in the following section, there is no “spatial wave packet” associated with the neutrinos, and the possibility (or not) of interference between the path amplitudes does not depend upon the existence (or not) of such a hypothetical spatial wave packet. The momentum wave packet referred to above is not related, by a Fourier transform, to any spatial wave packet with simple physical properties. The muon is an unstable particle with an exponential decay law and mean lifetime τ_μ .

The Fourier transform of this exponential yields, in energy space, a Breit–Wigner amplitude with width parameter $\Gamma_\mu = 1/\tau_\mu$ that describes the distribution of the physical mass, W_μ , of the muon. Using energy-momentum conservation, the corresponding spread in W_μ generates a spread in the momentum and energy of the neutrino. This spread, weighted by the Breit–Wigner amplitude, generates the momentum wave packet referred to previously. Taking the Fourier transform of the Breit–Wigner amplitude will give back the original exponential decay law, not a spatial wave packet.

The irrelevance of the momentum–space uncertainty relation to the problem under discussion is demonstrated by applying it to the neutrino momentum wave packet in pion decay just discussed. With $\Delta p_\nu = m_\mu \Gamma_\mu / m_\pi$ [3] the corresponding spatial uncertainty is $\Delta x_\nu = 1.27$ km. Does this give a limit on the possible knowledge of the spatial position of the neutrino? By measuring the time of decay of the pion through detection of the decay muon, with the easily obtained precision of 10^{-10} s, the subsequent position⁸ of the neutrino is known with an uncertainty of $c \times 10^{-10} = 3$ cm, since the speed of light is precisely known. The known uncertainty in the position of the neutrino is then a factor 4×10^4 smaller than Δx_ν as calculated from the uncertainty relation. The latter evidently has no relevance to the actual or possible knowledge of the spatial position of the neutrino. Except for the spread in decay times of the parent pion (described by the energy-time uncertainty relation) there is then no “quantum uncertainty” on the position of the neutrino. Once created, its position can be known with classical precision.

The final state in (67), in its application to flavour oscillations, is the spatial wavefunction of the detection event. At the microscopic level, this can be considered as localised to within an uncertainty given by the spatial wavefunction of the target particle, which is of atomic dimensions (say 10^{-8} – 10^{-7} cm). This, however, does not reflect the experimental knowledge of the position of the detection event, that is limited, by considerations of experimental resolution, to the group of states F in (67). In a typical experiment, the experimental resolution might be of the order of a centimetre, seven or eight orders of magnitude larger than the microscopic localisation of the target particle. Thus, to a very good approximation, the states $|f_m\rangle$ can be assumed to be spatial eigenstates. The correction for the actual distribution of detection events is then included by performing a suitable average over the oscillation probability, as the contribution of different final states is incoherent, requiring classical addition of probabilities. An exactly similar argument applies to the initial states $|i_l\rangle$ corresponding to the spatial position of the source particle at some well-defined time. These states also can be assumed, for calculational convenience, to be spatial eigenstates. It is important to stress that this *does not imply* that the momentum uncertainty of the source particle is infinite (inconsistent with the assumption that it is at rest). This is a gross misinterpretation of the meaning of

⁸ Its direction can be determined by simultaneous measurement of the direction of the decay muon

the momentum–space uncertainty relation. Consider the case that the source particle is bound in an atom. The actual quantum positional uncertainty, at the time chosen to define the initial state, could be included by writing its spatial wavefunction explicitly in the path amplitude. However this distance is so much smaller than the experimenter’s actual knowledge of the position of the source, which is all that matters to accurately evaluate P_{FI} , that it is evident that the same result will be obtained if, for simplicity, a spatial eigenstate is assumed for the source particle. In an analogous way, energy-momentum eigenstates are conventionally assumed for all initial and final state particles in the momentum space calculation of invariant amplitudes in quantum field theory. This in no way affects the correctness of the results obtained. A naive application of the momentum–space uncertainty relation, as done above for the path amplitude calculation, to this case would imply an infinite uncertainty on the space-time position of the scattering event or decaying particle. Actually the space-time parts of the wavefunctions of the incoming and outgoing particles play no role in a momentum space calculation except for imposing the constraints of exact energy and momentum conservation. If this is assumed *ab initio* the space-time parts of the wavefunctions can be omitted completely without changing the results of the calculations.

Of course, in any actual experiment, the source particle is not at rest. If it is negatively charged and forms an atomic bound state it will have some Fermi motion. If it is positively charged, and remains unbound, it will be subject to random thermal motion. Corrections for these effects can be included in the calculation of the oscillation frequency by a suitable averaging procedure. Since they affect only the definition of the group I of initial states, these corrections are incoherent, and, just as in the case of the positions of the source particle and detection event, the averaging must be done at the level of probabilities, not amplitudes. In fact averaging the source momentum distribution at the amplitude, rather than probability, level will result in the generation of a spurious “momentum wave packet” with parameters characteristic of the initial momentum *wavefunction*. As discussed in the following section, just such a mistake has been made in many papers on the QM of neutrino oscillations in the literature.

Before defining the intermediate states $|k_i\rangle$ in (67) for the case of lepton flavour oscillations, it is important to consider the nature of the decay process in which the neutrino mass eigenstates $|\nu_1\rangle$ and $|\nu_2\rangle$ are produced, as already mentioned in Sect. 1. In the standard electroweak model, these states are created by diagrams containing real or virtual W bosons. In the case that neutrinos are massive, and the different eigenstates have different masses, there is a non-diagonal, unitary, flavour/mass mixing matrix, the MNS [9] matrix, that is strictly analogous to the better-known CKM [10] quark flavour/mass mixing matrix. The existence of such a non-diagonal matrix implies that lepton (or generation) number is not conserved in transitions in the lepton sector just as generation number is not conserved in the quark sector because of the

non-diagonal nature of the CKM matrix. For example, the decays

$$\pi^+ \rightarrow (W^+)^* \rightarrow \mu^+ \nu_1, \quad \pi^+ \rightarrow (W^+)^* \rightarrow \mu^+ \nu_2, \quad (70)$$

where $(W^+)^*$ denotes a virtual W^+ boson, are strictly analogous to the quark sector processes, where an on-shell W^+ boson decays into quarks:

$$W^+ \rightarrow c\bar{s}, \quad W^+ \rightarrow c\bar{d}, \quad (71)$$

in which the first (second) transitions in (71) are Cabibbo allowed (suppressed). The neutrino mass/flavour mixing angle θ in Table 1 plays the same role in (70) as the Cabibbo angle θ_c in (71). Note that the μ^+ in (70) and the c quark in (71) are both second generation particles, whereas the decay antiquarks are either first or second generation. This suggests that the neutrino mass eigenstates should also be associated, as are the quark mass eigenstates, to different generations. Just as the W decays in (71) are *independent physical processes*, so also are the pion decays in (70). This has the important consequence, for the following discussion, that $|\nu_1\rangle$ and $|\nu_2\rangle$ may be produced at *different times* in the corresponding path amplitudes. As will be seen, this leads to an important new contribution to the oscillation phase. The incoherent nature, as in (70) above, of decays into different mass eigenstates has been previously pointed out by Shrock [7, 8].

What has been assumed until now in all discussions of the QM of neutrino oscillations in the literature is that lepton number is conserved in pion decay, so that the unique physical process is $\pi^+ \rightarrow \mu^+ \nu_\mu$. Thus the “flavour momentum eigenstate” $|\nu_\mu\rangle$ is assumed to be produced at a fixed time. It is then rewritten, at this time, in terms of $|\nu_1\rangle$ and $|\nu_2\rangle$ using the mixing amplitudes given in Table 1. These mass eigenstates subsequently evolve in time according to the different propagators discussed in Sects. 2, 3 and 4 above. The present writer’s opinion is that this procedure is wrong. If neutrinos of different generations have different masses, lepton number *cannot* be conserved at charged current weak vertices, just as, in the quark sector, generation number is not conserved at such vertices. The non-existence of a consistent theoretical description of “flavour momentum eigenstates” has been previously stressed in the literature [22]. Once the incoherent nature of, for example, the two processes in (70), is recognised, it will be seen that the conceptual problems that have beset the treatment of the QM of neutrino oscillations in the past, as evidenced by the many kinematical and geometrical inconsistencies discussed above, all disappear. The neutrino oscillation phenomenon occurs even when the constraints of both exact energy-momentum conservation and space-time geometry are imposed. The latter is simply the condition that the source particle is defined and the detection event observed, each at a unique space-time point, and that both the source particle and the neutrinos follow classical space-time trajectories that, together, link these two points.

The detailed application of (67) to the derivation of the oscillation probability for a simple two-flavour problem, with the source at rest, will now be discussed. A

specific example is given by “ $\nu_\mu \rightarrow \nu_e$ ”⁹ oscillations following the decay, at rest, of a π^+ . The initial state corresponds to the undecayed source particle, assumed to be at rest (or to have, in any case, a very small and random Fermi or thermal motion), at some arbitrary time t_0 . At later times t_1 or t_2 the source particle may decay into the eigenstates $|\nu_1\rangle$ or $|\nu_2\rangle$ respectively. These two possibilities are, of course, in classical physics, mutually exclusive. In QM however, as the two different decay modes are indistinguishable, when only the detection event is observed, the corresponding amplitudes, not probabilities, must be added. The final state corresponds to the detection event, which occurs at time t_D at distance L from the source, and which may be produced by an interaction of either of the (indistinguishable) neutrino mass eigenstates with a particle of the detector. Suppose now that $m_1 > m_2$. Since either neutrino mass eigenstate may create the detection event, and since $v_2 > v_1$, it follows that $t_2 > t_1$.

The different quantum states in (67) may now be identified. The initial state is $|S_l, t_0\rangle$ where S denotes the source particle. The label l corresponds to different source positions in the experimental apparatus, as well as, possibly, small random momenta of the source particle. The source wavefunction then propagates in time (but not in space) over the interval $t_0 < t < t_i$. This evolution is described, in the case of a source particle of mass m_S and decay width Γ_S , by the Feynman invariant space-time propagator [20, 21]

$$\langle S_l, t_i | S_l, t_0 \rangle = \exp \left[\left(-im_S - \frac{\Gamma_S}{2} \right) (\tau_i - \tau_0) \right] \quad (72)$$

(source particle of mass m_S)

where τ_i and τ_0 are the proper times in the rest frame of S corresponding to the laboratory times t_i and t_0 . For the case of a source which is a β -radioactive nucleus, where the decay process is more appropriately described by non-relativistic quantum mechanics, the source propagator is

$$\langle S_l, t_i | S_l, t_0 \rangle = \exp \left[\left(-iE_\beta - \frac{\Gamma_S}{2} \right) (\tau_i - \tau_0) \right], \quad (73)$$

(β decay of a nucleus)

where, neglecting the recoil energy of the daughter nucleus, E_β is the total energy release in the β decay process. The formula (73) may be derived in the same way as the analogous one for a radiative transition in atomic physics, by the use of perturbation theory and the time dependent Schrödinger equation [23]. The appropriate amplitude to describe the β decay transition is then the matrix element $\langle N_f | T | N_i \rangle$ where N_i and N_f denote the parent and daughter nuclei respectively. Since these are in energy eigenstates the Schrödinger equation predicts that

⁹ “ $\nu_\mu \rightarrow \nu_e$ ” is only a shorthand notation to indicate that a muon is involved in the production process and that an electron is observed in the detection event. This notation is used, without quotes below, as is conventional. It should, however, not be forgotten that ν_μ and ν_e do not exist as physical particles if neutrinos are massive

they evolve in time as $|N_j, t\rangle = |N_j, t_0\rangle \exp[-iE_j(t - t_0)]$ ($j = i, f$). It follows that

$$\begin{aligned} \langle N_f, t | T | N_i, t \rangle &= \langle N_f, t_0 | T | N_i, t_0 \rangle \exp[-i(E_i - E_f)(t - t_0)] \\ &\simeq \langle N_f, t_0 | T | N_i, t_0 \rangle \exp[-iE_\beta(t - t_0)], \end{aligned} \quad (74)$$

where, in the last member of (74), the recoil energy of the nucleus N_f has been neglected. In the path amplitude formalism, $\langle N_f, t_0 | T | N_i, t_0 \rangle$ may be considered as a time independent amplitude (the same in all interfering path amplitudes) while the exponential factor in (74) represents the propagator of the parent nucleus. It is assumed, for simplicity, in (74) that $t - t_0 \ll 1/\Gamma_S$. For a source at rest where $t = \tau$, (74) leads to (73). It is interesting to note that, if N_i and N_f in (74) are replaced, for pion decay, by the quark degrees of freedom, via the correspondences $|N_i\rangle \simeq |\pi\rangle$ and $|N_f\rangle \simeq |0\rangle$, where $|0\rangle$ denotes the vacuum (zero energy) state of the quark sector, then the rest frame pion propagator is predicted by (74) to be $\exp[-im_\pi \Delta\tau]$, in agreement with the Lorentz invariant Feynman space-time propagator of (72) above.

The source particle then decays into a final state containing ν_i with the time independent amplitude

$$\begin{aligned} A(i \leftarrow S) &= \langle i | \alpha \rangle \langle \alpha | T | S_l \rangle \\ &= \langle i | T | S_l \rangle. \end{aligned} \quad (75)$$

In the first line of (75), the neutrino mass/flavour mixing amplitude is shown explicitly. However, as explained above, like the elements of the CKM matrix, this amplitude is an intrinsic part of the charged current coupling, so that the amplitudes $A(1 \leftarrow S)$ and $A(2 \leftarrow S)$ describe independent physical processes. Thus there is no reason for the assumption $t_1 = t_2$ that has been made in essentially all discussions of the QM of neutrino oscillations in the literature to date.

The third element in the path amplitude is the invariant space-time propagator of the neutrino [20, 21]

$$\langle i, x_D, t_D | i, x_i, t_i \rangle = \exp[-im_i(\tau_D - \tau_i)]. \quad (76)$$

The fourth and last element in the path amplitude is the time independent amplitude of the detection process

$$\begin{aligned} A(d \leftarrow i) &= \langle d_m | T | \beta \rangle \langle \beta | i \rangle \\ &= \langle d_m | T | i \rangle, \end{aligned} \quad (77)$$

where the label m specifies the spatial position of the detection event, as well as possibly the directions and energies of the particles, in the final state of the event, that are detected.

The path amplitudes are then given by combining (72) or (73), (75), (76) and (77):

$$\begin{aligned} A(i \leftarrow \beta \leftarrow \alpha) &= \\ &\langle d_m | T | \beta \rangle \langle \beta | i \rangle \langle i, x_D, t_D | i, x_i, t_i \rangle \langle i | \alpha \rangle \langle \alpha | T | S_l \rangle \langle S_l, t_i | S_l, t_0 \rangle. \end{aligned} \quad (78)$$

The phase increment $\Delta\phi_i$ can be read off directly from (72) or (73) and (76):

$$\Delta\phi_i = m_i(\tau_D - \tau_i) + E_S(\tau_i - \tau_0), \quad (79)$$

where $E_S = m_S$ or E_β for unstable particle or nuclear β decay sources respectively. Now,

$$\begin{aligned} \tau_i - \tau_0 &= \frac{t_i - t_0}{\gamma_S} = t_i - t_0 + O(v_S^2) \\ &= t_D - t_0 - \frac{L}{v_i} + O(v_S^2) \\ &= t_D - t_0 - L \left[1 + \frac{m_i^2}{2p_0^2} \right] + O(v_S^2) + O(m^4), \end{aligned} \quad (80)$$

so that, using (10) and (80), (79) may be written (neglecting any random motion of the target) as

$$\Delta\phi_i = \frac{m_i^2}{p_0} \left[1 - \frac{E_S}{2p_0} \right] L + E_S(t_D - t_0) - E_S L + O(m^4), \quad (81)$$

giving for the oscillation phase

$$\phi_{12} = \frac{\Delta m^2}{p_0} \left[1 - \frac{E_S}{2p_0} \right] L + O(m^4), \quad (82)$$

where

$$\Delta m^2 \equiv m_1^2 - m_2^2. \quad (83)$$

It can be seen that, in addition to the contribution due to neutrino propagation, first given by Gribov and Pontecorvo (the first term on the RHS of (82)), there is also an important contribution due to the propagation of the source particle during the time interval $t_1 < t < t_2$. For example, in the case of pion decay at rest, $E_S/2p_0 = m_\pi/2p_0 = 2.34$, so that the oscillation phase is 34% larger than the Gribov and Pontecorvo result and a factor 2.68 times larger than the prediction of the standard formula. In order to calculate the oscillation probability, from flavour α to flavour β , at the fixed source–detector separation, L , the probability corresponding to the sum of the amplitudes in (78) must be integrated over the detection time t_D . For this, it is convenient to introduce the times-of-flight $t_1^{\text{fl}}, t_2^{\text{fl}}$ of the neutrinos:

$$t_i^{\text{fl}} = t_D - t_i = L \left[1 + \frac{m_i^2}{2p_0^2} \right] + O(m^4); \quad (84)$$

then

$$\begin{aligned} P_{\beta\alpha} &= \int_{t_{\min}}^{\infty} |A_1 + A_2|^2 dt_D \\ &= |\langle d_m | T | \beta \rangle \langle \alpha | T | S_l \rangle|^2 \\ &\times \int_{t_{\min}}^{\infty} e^{-\Gamma_S(t_D - t_0)} \left[(\langle \beta | 1 \rangle \langle 1 | \alpha \rangle)^2 e^{\Gamma_S t_1^{\text{fl}}} \right. \\ &+ (\langle \beta | 2 \rangle \langle 2 | \alpha \rangle)^2 e^{\Gamma_S t_2^{\text{fl}}} + 2 \langle \beta | 1 \rangle \langle 1 | \alpha \rangle \langle \beta | 2 \rangle \langle 2 | \alpha \rangle \\ &\times e^{\Gamma_S \left(\frac{t_1^{\text{fl}} + t_2^{\text{fl}}}{2} \right)} \cos \frac{\Delta m^2}{p_0} \left(\frac{E_S}{2p_0} - 1 \right) L \left. \right] dt_D. \end{aligned} \quad (85)$$

t_{\min} takes the values $t_0 + t_1^{\text{fl}}, t_0 + t_2^{\text{fl}}$ and $t_0 + t_1^{\text{fl}}$ for the squared amplitudes for neutrinos of masses m_1, m_2 and the interference term respectively, where it is assumed that $m_1 > m_2$, so that $t_1^{\text{fl}} > t_2^{\text{fl}}$. The t_D integral cancels the time-of-flight dependence of the squared amplitude terms and gives a factor $\exp[-\Gamma_S(t_1^{\text{fl}} - t_2^{\text{fl}})/2]$ multiplying the interference term. Thus, using (84) the following time-averaged oscillation probability is obtained:

$$\begin{aligned} P_{\beta\alpha} &= \frac{|\langle d_m | T | \beta \rangle \langle \alpha | T | S_l \rangle|^2}{\Gamma_S} \\ &\times \left[(\langle \beta | 1 \rangle \langle 1 | \alpha \rangle)^2 + (\langle \beta | 2 \rangle \langle 2 | \alpha \rangle)^2 \right. \\ &+ 2 \langle \beta | 1 \rangle \langle 1 | \alpha \rangle \langle \beta | 2 \rangle \langle 2 | \alpha \rangle \\ &\times \exp \left(-\Gamma_S \frac{\Delta m^2 L}{4p_0^2} \right) \cos \frac{\Delta m^2}{p_0} \left(\frac{E_S}{2p_0} - 1 \right) L \left. \right]. \end{aligned} \quad (86)$$

In all conceivable neutrino oscillation experiments the exponential damping of the interference term is very small. Since observation of the oscillation requires the argument of the cosine in (86) to be of order unity, i.e., $\Delta m^2/p_0 \simeq 1$, the damping term is typically $\simeq \exp(-\Gamma_S/4p_0)$. In the case of pion decay, where $\Gamma_S = 2.5 \times 10^{-14}$ MeV, and $p_0 = 29.8$ MeV, the damping factor is $\simeq 1 - 4.1 \times 10^{-16}$. For $\Delta m^2 = (1 \text{ eV})^2$, the distance L , in pion decay at rest, is 1.9×10^{16} m or 2.0 light yr for 50% damping of the interference term.

Two important special cases of (86) are $P_{e\mu}$, as, for example in $\nu_\mu \rightarrow \nu_e$ oscillations following π^+ decay (ν_e appearance) and P_{ee} as in $\nu_e \rightarrow \nu_e$ following β^+ decay using a nuclear reactor as a source (ν_e disappearance). Neglecting the exponential damping of the interference term, and using the flavour/mass mixing amplitudes in Table 1, the following predictions are obtained

$$\begin{aligned} P_{e\mu} &= 2 \frac{|\langle d_m | T | e \rangle \langle \mu | T | S_l \rangle|^2 \sin^2 \theta \cos^2 \theta}{\Gamma_S} \\ &\times \left[1 - \cos \frac{\Delta m^2}{p_0} \left(\frac{m_\pi}{2p_0} - 1 \right) L \right], \end{aligned} \quad (87)$$

$$\begin{aligned} P_{ee} &= \frac{|\langle d_m | T | e \rangle \langle e | T | S_l \rangle|^2}{\Gamma_S} [\sin^4 \theta + \cos^4 \theta \\ &+ 2 \sin^2 \theta \cos^2 \theta \cos \frac{\Delta m^2}{p_0} \left(\frac{E_\beta}{2p_0} - 1 \right) L]. \end{aligned} \quad (88)$$

The derivation of (86) has neglected velocity smearing of the neutrinos due to the finite decay widths of the source, and also possibly, the recoil particles. These effects have been estimated for the case of pion decay (using a Gaussian approximation for the Breit–Wigner amplitudes) in [3]. They are found to be more than ten orders of magnitude smaller than the already minute correction term for lifetime damping discussed above. Corrections to (86) due to finite experimental source and detector sizes, as well as the effect of thermal motion (for a positive pion source) are also calculated in [3].

An extended discussion of the application of the Feynman path amplitude method to flavour oscillations of neutral kaons and B^0 mesons will be presented elsewhere [24].

Here, only a few remarks on the salient differences with respect to neutrino oscillations will be made.

(i) Because of the very small fractional mass differences between different neutral kaon and B^0 meson mass eigenstates [25]:

$$\frac{\Delta m_K}{m_{K^0}} = \frac{m(K_L) - m(K_S)}{m_{K^0}} = 7.1 \times 10^{-15}, \quad (89)$$

$$\frac{\Delta m_B}{m_{B^0}} = \frac{m(B_H) - m(B_L)}{m_{B^0}} = 5.9 \times 10^{-14}. \quad (90)$$

The Lorentz- γ factors relating the proper time and laboratory times of the mass eigenstates are then equal to within the fractional differences quoted in (89) and (90). Thus, unlike for the case of neutrinos, the “equal velocity” hypothesis, discussed in Sect. 2 above, is expected to be a good kinematical approximation.

(ii) The unobserved propagating particles K_S , K_L and B_L , B_H are unstable, with decay widths that for K_S , B_L , and B_H are of the same order as the mass differences:

$$\frac{\Gamma_S}{\Delta m_K} = 2.10, \quad (91)$$

$$\frac{\Gamma_L}{\Delta m_K} = 0.037, \quad (92)$$

$$\frac{\Gamma_{B_L}}{\Delta m_B} = \frac{\Gamma_{B_H}}{\Delta m_B} = 1.37. \quad (93)$$

Unlike in the case of neutrino oscillations, where the natural widths of the source and recoil particles give only very small variations of velocity, the decay widths of the K_S , B_L and B_H then produce velocity smearing effects of the same order as the velocity differences resulting from the mass differences. Such effects must then be taken properly into account in the calculation of the flavour oscillation probability. However, because $\Gamma_L \ll \Gamma_S$, the natural width of the K_L may, in first approximation, be neglected. Including correctly the kinematical effect of the physical mass, W , of the propagating particles modifies the “on-shell” expression (10) for the phase increment to

$$\Delta\phi_i = m_i \Delta\tau_i = \frac{m_i \Delta t_i}{\gamma_i} = \frac{m_i W_i \Delta t_i}{E_i} = \frac{m_i W_i L}{p_i}. \quad (94)$$

It is important to note that although, for example, the K_S is significantly off-shell on the scale of Δm_K , the condition $c\Delta\tau \gg 1/W_S$ is still well satisfied in a typical flavour oscillation experiment, so the off-shell K_S can still be considered to propagate as a classical particle according to (94). It may also be remarked that the pole mass, m_i , in (94) results from the Fourier transform of the invariant momentum space propagator, and so is constant for each particle species, unlike the variable physical mass W_i .

(iii) Unlike neutrinos which, in terrestrial oscillation experiments, are produced incoherently as single quantum states from a coherent source (an unstable particle or nucleus) neutral kaons may either be produced incoherently, in an incoherent interaction, such as $\pi^- p \rightarrow \Lambda(K_L, K_S)$ or $K^- p \rightarrow n(K_L, K_S)$, or in correlated “entangled” states,

from a coherent source, as in $\phi \rightarrow K_L K_S$. To date, B^0 mesons have usually been produced in entangled states from coherent sources, for example $Z \rightarrow B_L B_H X$ or $\Upsilon(4S) \rightarrow B_L B_H$.

Because of the points (ii) and (iii) above, the path amplitude analysis of K_S - K_L and B_L - B_H oscillations is more complicated than for the case of neutrino oscillations. The standard procedure, to date, in analysing experiments, is to assume equal velocities (that seems in view of (89) and (90) to be reasonable) so that, for example, $\Delta\tau_L = \Delta\tau_S = \Delta\tau$ and use the first member of (94), together with the relation $\Delta\tau = m_{K^0} L / p_{K^0}$ to give the formula

$$\Delta\phi_i = \frac{m_i m_{K^0} L}{p_{K^0}}. \quad (95)$$

Setting $m_{K^0} = (m_L + m_S)/2$ in (95) gives, for the oscillation phase:

$$\begin{aligned} \phi_{LS} &= \frac{\Delta m_K m_{K^0} L}{p_{K^0}} = \frac{(m_L - m_S)(m_L + m_S)L}{2p_{K^0}} \\ &= \frac{(m_L^2 - m_S^2)L}{2p_{K^0}}. \end{aligned} \quad (96)$$

This is the same as the standard formula for neutrino oscillations, also derived on the basis of the equal velocity hypothesis. At the time of writing, the author is not able to comment on the correctness or otherwise, of (96).

Taking into account exact energy-momentum conservation in the process $\pi^- p \rightarrow \Lambda(K_L, K_S)$, but assuming all particles to be on-shell, the authors of [26] obtained a different formula to (96), and they predicted as well correlated spatial oscillations in the decays of the neutral kaons and the lambda.

6 Discussion

There are two essential differences between the work presented in the present paper and [3, 6], and all previous treatments of the QM of neutrino oscillations in the literature:

- (i) the realisation that the different neutrino mass eigenstates are produced in independent physical processes, not as a coherent lepton flavour eigenstate, and
- (ii) the assumption that, in the covariant Feynman path amplitude description, the neutrino mass eigenstates follow essentially classical space-time trajectories; there is no spatial “wave packet” associated with their propagation.

Historically, as will be seen, the failure to notice (i), leading to the universal assumption that the different neutrino mass eigenstates are produced coherently and at the same time, required the introduction of a spatial wave packet, since otherwise the neutrino mass eigenstates, produced at the same time and propagating with different velocities could not produce the detection event at a well-defined space-time point. Only, it was argued, by introducing the spatial “fuzziness” associated with the hypothetical spatial wave packet would interference be possible. In the Feynman path description there is no such

spatial “fuzziness”. The detection event can be produced by the interaction of either (indistinguishable) neutrino mass eigenstate with the target particle in the detector. Since the space-time trajectories of the neutrinos are essentially classical, with different velocities, this implies different production times for the neutrinos in the alternative interfering amplitudes. In view of (i) and the long lifetimes of typical neutrino sources different decay times are allowed. This is the fundamental reason for the possible occurrence of the oscillation phenomenon. The propagator of the source particle contributes to the path amplitudes associated with each neutrino mass eigenstate, and, as a consequence of the different decay times, gives an important contribution to the interference phase, as demonstrated by the calculations presented in Sect. 5 above.

The important point (i) above was pointed out many years ago by Shrock [7, 8] but was never, to the author’s best knowledge, applied before the work reported in this paper and [3, 6] to the neutrino oscillation problem. Indeed, the contrary, incorrect, hypothesis of an initial “lepton flavour eigenstate”, that is a coherent superposition of different mass eigenstates, has, instead, been universally assumed in the treatment of neutrino oscillations.

A brief historical review of the introduction of the “wave packet” concept into the description of neutrino oscillations will now be made. The first descriptions of neutrino oscillations were based on plane wave propagators either (tacitly) covariant in the case of Gribov and Pontecorvo [1], or, temporal only, as in the case, for example, of Fritsch and Minkowski [2] who were the first to derive the standard oscillation phase. The wave packet concept in the context of neutrino oscillations was first introduced by Nussinov [27], together with the idea of a “coherence length” for neutrino oscillations. Nussinov discussed the effect analogous to collision broadening of atomic spectral lines for neutrino sources in the sun. Both the “wave packet” and “coherence length” concepts are related to a classical wave rather than a quantum mechanical description of the associated phenomena. A source of classical waves of finite duration will evidently produce a wave-train of finite length. Fourier analysis of this spatial wave-train will result in a spread in the momentum of the associated plane waves described by a coherent momentum wave packet. This classical description does not match the sequence of spatio-temporal phenomena underlying the line broadening effect as described by quantum mechanics. Actually, the unstable source produces particles (photons or neutrinos) in a process that has a characteristic time¹⁰ much shorter than the mean decay time of the source. Interference effects will occur if some process, initiated by a decay particle, can correspond to different production times of the latter. In the case of, say, atomic line broadening as observed in the interference-fringe contrast in a Michelson interferometer, this time difference results from the different propagation times of the photon in the different arms of the interferometer. In the case of neutrino oscillations, with a source at rest, it results

from different times-of-flight of the different mass eigenstates over the same spatial distance. In both cases the line broadening effect results from perturbation of the coherent source during the time interval between the two emission times corresponding to the quantum interference condition. Thus the fundamental physical parameter governing the interference damping is the effective lifetime of the coherent source, due either to its spontaneous decay rate or to collision processes. Nussinov correctly identified this parameter, τ_{eff} , as the one controlling the damping of the oscillations, but then introduced a hypothetical “spatial wave packet” of length $c\tau_{\text{eff}}$. This is actually a classical wave analogue of the series of spatio-temporal quantum processes just described. As is well known from atomic radiative transitions the classical wave and QM calculation both lead to a Lorentzian line shape and so are equivalent if only a description of the momentum distribution of the produced particles is required¹¹. As shown in the calculations in Sect. 5, a correct description of interference damping requires the spatio-temporal sequence of quantum events to be properly taken into account. This is not possible in the classical wave description in which a hypothetical spatial wave packet is introduced. It must be stressed that in the quantum mechanical description of atomic radiative transitions there is no wave packet associated with photon propagation. The Fourier transform of the exponential decay law of the initial atom produces a Lorentzian distribution in the energy of this atom. When it decays by photon emission, energy conservation then produces a smearing in the energy (or wavelength) of the photon that reflects the energy uncertainties of both the initial and final atoms. But the photon propagates in space-time like a particle with velocity c . There is no associated spatial, or momentum, wave packet. Indeed, a photon is a particle and not a classical wave.

Krauss and Wilczek [29] who also estimated the effect of collision broadening on solar neutrino sources, although also mentioning the wave packet description, understood that neutrino oscillations require different emission times in the interfering amplitudes: “Since these mass eigenstates propagate with different velocities (for fixed energy) the desired interference is between neutrinos emitted at *different times*.” Actually, the mass eigenstates do not have fixed energy (or momentum) but certainly have different velocities, so that the above assertion concerning the interference mechanism underlying neutrino oscillations is correct.

The first extended discussion of the QM of neutrino oscillations in terms of wave packets was made by Kayser [30]. In contradiction with the conclusions of Shrock published in the previous year [7] the initial state of the neutrinos was assumed to be a superposition of mass eigenstates with definite lepton flavour, and the propagating neutrinos

¹⁰ Possible ways to experimentally measure such a “process duration” time are discussed in [28]

¹¹ This is only true as regards the mathematical form of the line shape. The width of the distribution predicted by the QM calculation depends on the lifetimes of both the initial and final states, so there is in general no direct relation between the line width and the length, $c\tau_{\text{eff}}$, of the analogous classical wave packet

were assumed to have equal momenta and different energies. Thus energy-momentum conservation is violated in the neutrino production process. Kayser claimed that, in order for the neutrino oscillation phenomenon to be possible, the uncertainty in the momentum of the neutrinos must satisfy the condition

$$\frac{\Delta p_\nu}{p_\nu} \geq \lambda \frac{\Delta m^2}{p_\nu^2}, \quad (97)$$

where $\lambda \simeq 10\text{--}100$. It was then proposed to realise this condition by introducing a hypothetical coherent momentum wave packet with a width consistent with the condition (97). This condition was derived from the momentum–space uncertainty relation as follows: It was assumed that the uncertainty, Δx_ν , in the position of the neutrino is related to its spread in momentum, Δp_ν , by the relation $\Delta x_\nu \Delta p_\nu \geq 1$. The requirement $\lambda \Delta x_\nu = \ell_{\text{osc}}$, where ℓ_{osc} is the neutrino oscillation length, then leads to (97), it being assumed that if $\Delta x_\nu \simeq \ell_{\text{osc}}$ it will be impossible to observe neutrino oscillations. Although this latter condition is evidently correct if Δx_ν refers to the experimental uncertainty in the observed position the neutrino detection event, it is easily shown to be completely false if, instead, it refers to the theoretical uncertainty calculated using the momentum–space uncertainty relation. For example, in the case of pion decay at rest the width of the coherent momentum wave packet associated with the spread in physical mass of the decay muon is $\Delta p = m_\mu \Gamma_\mu / m_\pi = 2.3 \times 10^{-16} \text{ MeV}$. Taking $\lambda = 10$, $\Delta m^2 = (1 \text{ eV})^2$ and $p_\nu = 29.8 \text{ MeV}$, (97) gives the limit $\Delta p_\nu / p_\nu > 1.1 \times 10^{-14}$ as compared to the width of the coherent momentum wave packet in pion decay at rest of $\Delta p_\nu / p_\nu = 7.2 \times 10^{-18}$. This is three orders of magnitude lower than Kayser’s lower limit for the possibility of neutrino oscillations, (97). Yet explicit calculation of the corresponding damping of the interference term [3] shows it to be completely negligible. As discussed in [3] the momentum spread of the neutrinos due to smearing of the physical mass of the decaying pion, or its thermal motion, are much larger than that associated with the physical mass of the decay muon. The two former sources are, however, incoherent, and so have no associated wave packets.

In fact, Kayser assumes that the mass eigenstates are both produced and detected at the same times, i.e. the equal velocity hypothesis is made, in contradiction with the different velocities resulting from the equal momentum and unequal energy hypothesis. This procedure is justified by the hope that the effect of this logical inconsistency will be annulled by the space-time fuzziness introduced by the hypothetical wave packet: “The wave packet treatment eliminates the need to make some idealising assumption by taking both momentum and energy variations properly into account”.

The “idealising assumption” referred to is actually the exact conservation of energy and momentum that is automatic in all covariant calculations of decay or scattering processes. The real purpose of the wave packet is, however, rather to enable the neutrinos, incorrectly assumed

to be always produced at the same time, but separating spatially due to their different velocities, to be detected at the same time. This is clearly impossible when, as is in fact the case, they move along essentially classical space-time trajectories.

Like Nussinov, Kayser introduces a hypothetical spatial wave packet of length $c\tau$ where τ is the lifetime of the decaying state. This wave packet, which, as explained above, is only a classical wave theory analogue of the finite source lifetime, does not exist in the QM calculation. The Fourier transform of the exponential decay amplitude yields a Breit–Wigner amplitude describing the distribution of the physical mass of the unstable source particle. As discussed in the previous section, as this mass is a property of the source particle, any resulting momentum smearing of the produced neutrinos is then an incoherent effect. The only coherent momentum smearing (yielding a momentum, not a spatial, wave packet) is that associated with the physical masses of any unobserved, unstable, recoil particles. Kayser then argues that the length of the spatial wave packet must be much shorter than $c\tau$: “If we are interested in a neutrino emitted at time $t = 0$, but we can learn only that the emitter was somewhere in a region of length h , then the amplitude for the emission to have occurred at $t = 0$ at the various points in this region must be added coherently. Thus the neutrino wave packet will have a length $d \simeq h$.”

The false assumption is made that both neutrinos must be created at the same time, $t = 0$. It is then assumed, in contradiction with (67) above, for the case of a neutrino source at rest, that the amplitudes of neutrinos created at different spatial positions must be added coherently. It corresponds to performing the sums over m and l in (67) in a coherent manner, i.e. with $\sum_m \sum_l$ inside the modulus squared. An extended criticism of this assumption may be found in [3]. In Kayser’s interpretation the spatial wave packet is needed to “delocalise” the neutrinos, that, because of the assumption of a common production time, become spatially separated due to their different velocities. In fact, the neutrinos can arrive at the detector at the same time because they may be produced at different times in the alternative histories corresponding to the different path amplitudes. There is then no need for the hypothetical spatial wave packet introduced by Kayser. There are no physical parameters in the QM calculation governing the size of such a wave packet. It does not exist except in an analogous classical wave theory calculation. The space-time structure of particle propagation cannot be described by such a calculation.

Because equal velocities are assumed, i.e. that both neutrinos are produced at one space-time point and both detected at another, Kayser obtains the standard oscillation phase, a result shown to be unmodified [30] by convolution with a narrow momentum wave packet.

Ten years after Kayser’s paper on wave packets in neutrino oscillations the first of many papers where spatial wave packets, as suggested by Kayser, were implemented using Gaussian functions, was written by Giunti, Kim and Lee [12]. This paper starts with the statement “If neutri-

nos are massive particles and mixed, a flavour neutrino is created by a weak interaction process as a coherent superposition of mass eigenstates.”

This statement is in contradiction with the findings of Shrock [7,8] that, if non-degenerate massive neutrinos exist, lepton flavour is not conserved by the weak interaction, so that the different mass eigenstates are produced incoherently in different physical processes. It implies that, as previously assumed by Kayser (and all subsequent authors of papers on the QM of neutrino oscillations), all neutrinos are produced at the same time. Therefore they must satisfy the equal velocity hypothesis of Sect. 2 above, and independently of the introduction, or not, of spatial wave packets, or the assumption, or not, of exact energy-momentum conservation in the production process, the standard result for the oscillation phase must be obtained.

The introduction of [12] contained a list of issues that it was claimed should be addressed in order to provide a “complete” treatment of the QM of neutrino oscillations. It is instructive to review this list from the perspective of the work presented in the present paper and [3,6]. The four issues were as follows.

- (i) “A necessary condition for neutrino oscillations to occur is that the neutrino source and detector are localised within a region much smaller than the oscillation length; then the neutrino momentum has at least the corresponding spread given by the uncertainty principle [30].”
- (ii) “The energy-momentum conservation in the process in which the neutrino is created implies that different mass eigenstate components have different momenta as well as different energies [14].”
- (iii) “The different mass eigenstates must be produced and detected coherently; this is possible only if the other particles associated with the production and decay processes have energy-momentum spreads larger than the energy-momentum differences of the mass eigenstates.”
- (iv) “The wavefunction of the propagating neutrino must be a superposition of the wavefunctions of the mass eigenstates with proper coefficients given by the amplitudes of the processes in which the mass eigenstate neutrinos are produced.”

The following comments (C) may be made on these points:

- (Ci) The first sentence is trivially correct, but is unrelated to the quantum mechanical aspects of the problem. The last phrase, based as it is on the arguments of Kayser (see (97)) is, as argued above, demonstrably without physical foundation.
- (Cii) This is indeed an essential ingredient of a correct treatment, in QM, of neutrino oscillations. A corollary is that the neutrinos, propagating over macroscopic distances, do so as classical particles [21]. This condition is violated by the universal equal velocity assumption of (17).
- (Ciii) Properly interpreted, the first sentence is correct. It means that the amplitudes describing the temporal sequence of events: (a) propagation of the source, (b) decay of the source, (c) propagation of a neutrino mass eigenstate and (d) production of the detection event must be added coherently. It does not mean that the mass eigen-

states are part of a coherent “flavour eigenstate” in either the detection or production processes. The last phrase, as shown explicitly for the case of pion decay at rest in Sect. 5 above, is false.

(Civ) The “wavefunction” referred to here is presumably a lepton flavour eigenstate propagating in space-time. As previously pointed out by Shrock [7,8] and shown later by Giunti, Kim and Lee themselves [22], such a state does not exist.

The effect of the Gaussian wave packet introduced in [12] on the oscillation phase is described in Sect. 3 above. For a critical discussion of the “spatial coherence length” and “momentum damping factor” generated by such wave packets see [3]. The point of view, explained there, of the present author is that, as the coherent spatial wave packet from which they are derived does not exist, they also are without any physical foundation. The oscillations are damped by the source lifetime, as shown in (86), and also by the width of the coherent momentum wave packet associated with the different possible physical masses of unobserved recoil particles produced in the production process. As shown in [3], both effects are expected to be vanishingly small in any foreseeable neutrino oscillation experiment.

An influential paper on the QM of heavy quark flavour oscillations was written by Lipkin [11]. Similar ideas were applied to neutrino oscillations in [15] and in an unpublished pre-print [16]. The starting point of [11] was the correct observation that all experiments measuring flavour oscillations actually observe only a spatially varying interference effect. This implies that all decay and detection times should be integrated over in order to derive the quantum mechanical probability to be compared with experiment. This is done, for example, in the derivation of (86) above. Lipkin interpreted this correct statement about the nature of the experiments as implying that time should not appear at all in the theoretical description of the experiments. Formulae containing the time were referred to as describing “non-experiments”. In all three papers cited above the initial state is incorrectly assumed to be in a pure flavour eigenstate that is a superposition of mass eigenstates. Thus, for example the K_S and the K_L or the different neutrino mass eigenstates are assumed to be produced at the same time. Since they must be detected at the same time, as only one detection event is observed, the equal velocity hypothesis (17) is thus assumed. Since the additional assumption of equal energies and different momenta of the particles is made (of course in logical contradiction with the equal velocity hypothesis) the temporal part of the Lorentz invariant plane wave does not contribute to the oscillation phase. The latter is then entirely determined by the spatial part, as described in Sect. 4 above. The standard result for the oscillation phase is then obtained. The arguments given by Lipkin to justify the choice of equal energies and different momenta of the propagating mass eigenstates are unconvincing. On the process $K^- p \rightarrow \bar{K}^0 n$ Lipkin states “Energy conservation requires the \bar{K}^0 to have a definite energy. When it is split into K_L and K_S components with dif-

ferent masses the two states have the same energy but different momenta". There would seem to be no physical justification for this apodictic statement. Momentum conservation requires the \bar{K}^0 to be produced with a definite momentum. Why should it not then "split" into two states with the same momentum, and different energies? Later Lipkin gives an argument based on non-relativistic kinematics ((2b) of [11]) as applied to the process $K^- p \rightarrow \bar{K}^0 n$ to justify the neglect of the energy difference between the K_L and K_S . Repeating the calculation using relativistic kinematics, as more appropriate to typical experimental conditions, shows instead that $\Delta p/p = \Delta E/E$, so that there is, in this case, no kinematical justification for the equal energy hypothesis. Actually the K_L and K_S , like the different neutrino mass eigenstates, are produced incoherently, in different physical processes, so that there is no production of the state " \bar{K}^0 ". Conservation of energy and momentum then shows that they must have different energies, different momenta and different velocities.

In [11] a calculation of the oscillation phase for the $B^0 \bar{B}^0$ system is performed in the laboratory system using temporal evolution, and allowing different propagation times for the different mass eigenstates ((9b) of [11]). As discussed in Sect. 4 above, this corresponds to the full Lorentz invariant phase in the non-relativistic limit where $p_i \ll m_i$. In this limit the complete $O(m^2)$ Gribov–Pontecorvo result, with the oscillation phase a factor two larger than the standard result, is obtained. Lipkin noticed this difference but rejected the correct result given, in the appropriate kinematical limit, by his (9b) on the grounds that, as the time appeared explicitly in its derivation, it corresponded to a "non-experiment".

In [15], similar arguments are applied to the neutrino oscillation case. Again, equal energies and different momenta and the (contradictory) equal velocity hypotheses are assumed, leading to the standard oscillation phase. The initial state is required to be a superposition of different mass eigenstates with pure flavour. For example, in the case of say $\pi^+ \rightarrow \mu^+ \nu$, the probability of detecting a ν_e is zero, and of detecting a ν_μ is unity, at $L = 0$. These conditions are used to fix the coefficients of the mass eigenstate superposition at $L = 0$. However, as is shown by inspection of (87) above, exactly the same boundary condition is respected by the result of the Feynman path amplitude calculation where the π^+ decays incoherently into the different mass eigenstates at different times and no unphysical "flavour eigenstate wavefunction" is introduced. It was claimed in [15] that the energy-momentum and space-time descriptions of flavour oscillations are "complementary" and that including them both leads to "double counting" of the oscillation phase by a factor of two. In fact, the Lorentz invariant oscillation phase contains (except in the rest frame of the propagating particle) both energy-momentum and space-time contributions that must both be included to obtain the correct result. The detailed considerations of Sects. 2–4 above show that attempts to use only energy-momentum or space-time descriptions in inappropriate kinematical

regions leads instead to "half counting" of the correct Lorentz invariant phase.

In [16], Lipkin justified the "equal energy" hypothesis by reference to a paper by Stodolsky [31] which attempted a non-relativistic density matrix description of flavour oscillations. It was proposed to use "stationary" beams of fixed energy to describe the system of propagating mass eigenstates. In this way the introduction of spatial wave packets was avoided. The present writer's opinion is that such an approach is ill-founded. In fact, the propagating particles describe classical trajectories, the detailed spatio-temporal structure of which is essential for the correct QM description of the phenomenon. This information is not available in the non-relativistic density matrix approach, which, in any case, is not appropriate to describe ultra-relativistic neutrinos.

The "stationary" source and target description with equal energies and different momenta for the neutrinos as well as Gaussian spatial wave packets for both the source and detector was used more recently by Ioannisian and Pilaftsis [32]. As the equal velocity assumption (17) was also made the standard oscillation phase was obtained.

It is interesting to note that, in an earlier paper, written together with Kayser [33] Stodolsky proposed a covariant Feynman path amplitude approach, akin to that of the present paper and [3, 6], to the description of "entangled" systems such as $\phi \rightarrow K_S K_L$. In this case, in contradiction to [31], and as previously assumed by Kayser [30], equal momenta and different energies were proposed. The treatment of [33] differs from that of the present paper and [3, 6] in that the equal velocity hypothesis (17) was assumed, so that the standard oscillation formula was obtained and the contribution to the oscillation phase of the coherent source was neglected. Also, unphysical spatial wave packets were introduced and the potentially important velocity smearing effects due to variations of order Δm_K in the physical mass of the K_S , as discussed in Sect. 5 above, were neglected. The present writer is in agreement with the main conclusion of this paper, that the "collapse of the wavefunction" often discussed in connection with entangled states, is only the collapse of a mathematical abstraction, that is actually irrelevant to the QM description of the experiment. It is perhaps, however, going a little too far to state, as in the last sentence of [33] that "The best answer, finally to the "question of the collapse of the wavefunction" is that there is no wavefunction." Indeed, the QM of flavour oscillations is better described in terms of Feynman path amplitudes, as proposed in [33]. However, the wavefunction does remain an important and useful concept in the description of "static" bound systems such as the hydrogen atom.

Following Rich [34] several authors [35–37] have used a non-relativistic Wigner–Weisskopf-type formalism to describe the complete production-propagation-detection process. The propagating virtual neutrinos are assigned the same energy and different momenta. In all cases the equal velocity assumption is made leading to the standard oscillation phase. The present writer's opinion is that such treatments take properly into account neither the ultra-

relativistic nature of the propagating neutrinos nor the sequence of spatio-temporal production and detection events necessary for a correct calculation of the oscillation phase. In one paper using this non-relativistic approach [37], the effect of the source lifetime was discussed. Although it was correctly concluded that the damping effect due to the finite source lifetime is negligible, it was proposed that the characteristic momentum spread in the Gaussian wave packet, related to the source, for muon decay at rest (actually the Fourier transform of a Gaussian spatial wave packet as proposed by Giunti, Kim and Lee [12] following the suggestion of Kayser [30]) should be $\simeq 10^{-3}$ MeV. This is the typical momentum due to thermal motion at room temperature. As discussed above, any momentum smearing due to this source or to the physical mass of the decaying muon is incoherent and not associated with any wave packet. Indeed, in the case of muon decay, since all recoil particles are stable, then, unlike in the case of pion decay, there is no momentum wave packet associated with neutrino propagation.

The equal energy, different momentum, hypothesis was also made by the authors of [38]. This was justified by assuming that both production and detection of neutrinos resulted from inelastic scattering on an infinitely heavy target. Although such an assumption guarantees the kinematical correctness of the equal energy hypothesis, it evidently does not correspond to actual neutrino experiments where the neutrinos are produced by the decay of an unstable source, and where energy-momentum conservation always requires (see (11) and (12) above) that both momenta and energies and, hence the velocities, are different. The equal velocity hypothesis (in contradiction with the different momenta of the neutrinos) was also made so that the neutrino propagator is purely spatial, giving, as shown in Sect. 4 above, the standard oscillation phase.

Many of the features of the covariant path amplitude calculation of Sect. 5 above and [3, 6] have been previously introduced into the discussion of neutrino oscillations. For example, the Lorentz invariant Feynman propagator for the neutrinos has been used in [21, 32, 39–42]. The only author to introduce explicitly the invariant propagator of the source particle was Campagne [39]. As the equal velocity hypothesis was also made there is no contribution to the oscillation phase from this propagator and the standard result was obtained for the oscillation phase.

The authors of [43] recognised that the derivation of the standard formula required the equal velocity hypothesis, and that if exact energy-momentum conservation is imposed, so that the neutrinos have different times-of-flight, an oscillation phase a factor of two larger is obtained. In spite of noting that neutrinos of widely differing masses, as expected theoretically, cannot have equal velocities, the use of the equal velocity hypothesis was, nevertheless, recommended. A short note of Okun and Tsukerman [44] pointed out the kinematical impossibility of the equal velocity assumption for neutrinos with different masses.

A paper by Giunti [45], considering the analogy between the interference effects in the Young double slit ex-

periment and in neutrino oscillations, claimed to demonstrate that the extra factor of two in the oscillation phase, due to neutrino propagation, obtained when the different neutrino velocities are correctly taken into account [3, 43], is incorrect. This argument was based on the correct observation that in the Young double slit experiment with photons, photon propagation gives no contribution to the interference phase. That this must be so is evident by setting $m_i = 0$ in (76) above. Giunti then claimed to have rejected the different velocity hypothesis [43] and the path amplitude calculations of [3] by *reductio ad absurdum* since, in the analogous Young double slit experiment, a vanishing interference phase, clearly excluded by experiment, is apparently predicted. However, Giunti neglects the contribution to the interference phase of the coherent source (the excited atom that produces the photon). The contribution of the source to the interference phase is, using the formula analogous to (73) above, for an atomic radiative transition:

$$\Delta\phi^{\text{source}} = E^*(t_1 - t_2) \simeq E_\gamma(t_1 - t_2) = E_\gamma(r_2 - r_1), \quad (98)$$

where E^* , and E_γ are respectively the atomic excitation energy and the photon energy, while t_1, t_2 and r_1, r_2 are defined in [45]. This formula is identical to (8) of [45], and since the contributions from the photon propagators vanish, the path amplitude calculation gives the usual result obtained in the classical wave theory of light. Having wrongly concluded that the photon is produced at a unique time, Giunti introduces a hypothetical Gaussian spatial wave packet (unphysical, because it has no relation to the physics of the photon production process) and demonstrates that the space-time smearing provided by the wave packet allows one to recover the same interference phase as in (98) above. Also the interference term is found to be damped by a factor dependent on the length of the wave packet. As discussed in [3], as well as above in the present paper, this damping factor, derived from the spurious spatial wave packet is also without any physical foundation. The damping of the interference term is actually produced by the finite lifetime of the coherent source that limits the value of $t_1 - t_2$ in (98).

A recent paper by De Leo, Nishi and Rotelli [46] has also considered the effect on the oscillation phase of different kinematical assumptions. As in [43] it was realised that only in the case of the equal velocity hypothesis is the standard oscillation phase obtained; in any other case the phase is a factor of two larger. This agrees with the conclusions of Sects. 2, 3 and 4 of the present paper. The difference between the kinematical discussions of the present paper and those of [46] is that in the former case, in accordance with the experimental conditions, and as previously pointed out in [11], a constant distance, L , is assumed between the source and the detection event. In the latter case this distance is allowed to vary, and also different creation times are allowed for the mass eigenstates. In the general kinematical discussion of Section II of [46], which may be compared to Sect. 2 above (since both discuss the Lorentz invariant oscillation phase) it is assumed that Δt is the same, but that Δx is different for the different neutrinos.

This is in complete disagreement with the experimental conditions of typical neutrino oscillation experiments. In order to permit different source–detector separations, a Gaussian spatial wave packet was introduced. It was concluded that by a suitable choice of creation times a pure flavour eigenstate can be obtained at creation. Indeed the necessary existence of such a state is the initial hypothesis on which the arguments given in the paper are based. It may be commented that, firstly, as pointed out long ago by Shrock [7, 8] no such flavour eigenstate exists since the neutrinos are created in separate, incoherent, processes, and secondly, the distance L must be constant, since the source is assumed to be at rest. Finally, although it is true that the different neutrinos may be created at different times, the spatial wave packet introduced to allow the possibility of different source–detector distances does not, as argued above, have any physical basis.

A very recent paper by Beuthe [42] makes the same basic assumptions as an earlier paper by Giunti et al. [47]. Both equal momentum and equal energy hypotheses are considered, but the equal velocity condition (17) is always assumed so that the standard oscillation phase is always recovered, and the important contribution to the oscillation phase of the coherent source particle is neglected. There is a lengthy discussion of the effects of hypothetical Gaussian wave packets used to describe both the source and detector particles.

Further critical discussion of different treatments in the literature of the QM of neutrino oscillations may be found in [3]. The same paper also describes briefly two atomic physics experiments, the “quantum beat” experiment [48] and the “photodetachment microscope” [49–51] where the Feynman path amplitude description has been successfully tested in experiments where spatially varying interference effects, very similar to particle flavour oscillations, have been observed.

7 Summary and outlook

The kinematical and geometrical discussion of Sects. 2, 3 and 4 above shows that the standard formula (5), for the contribution of neutrino propagation to the oscillation phase, is a consequence of the equal velocity hypothesis where it is assumed that both interfering neutrinos are produced at the same space-time point. This hypothesis is incompatible with the propagation of the neutrinos along classical space-time trajectories if they have different masses. The oscillation phase, calculated at $O(m^2)$, using the equal velocity hypothesis is found to be a factor of two smaller than the result first obtained by Gribov and Pontecorvo [1]. The latter is obtained by imposing both exact energy-momentum conservation and a consistent geometrical propagation in space-time, taking properly into account the different neutrino velocities. It is referred to above as the “exact” $O(m^2)$ formula. Thus the standard formula neglects numerically important $O(m^2)$ terms as compared to the exact one.

This conclusion remains the same whether the neutrino propagator is described in a Lorentz invariant man-

ner using plane waves (Sect. 2), whether convolution with a Gaussian wave packet is performed (Sect. 3) or whether only temporal or spatial evolution of the neutrino wavefunction is considered (Sect. 4). In contrast, the assumptions of equal momenta and different energies or of equal energies but different momenta give only negligible $O(m^4)$ corrections to the exact formula, or to the standard formula, as compared to the phase calculated assuming exact energy-momentum conservation.

The Feynman path amplitude calculation of Sect. 5 is based on the use of the exact Lorentz invariant plane wave formula (or, equivalently, the invariant Feynman space-time propagator) for the contribution of neutrino propagation to the oscillation phase. Since the different neutrino mass eigenstates are produced in independent physical processes [3, 7, 8] the decay of the source can occur at different times in the amplitudes describing the propagation of different neutrinos. The oscillation phenomenon then occurs respecting the constraints of both exact energy-momentum conservation and exact space-time geometry. The different decay times of the source in the interfering amplitudes then lead to an important contribution to the oscillation phase from the space-time propagator of the source.

Damping of the interference term due to the finite source lifetime or momentum smearing related to the off-shell nature of the source or recoil particles has been previously found to be completely negligible in all foreseeable neutrino oscillation experiments [3].

All previous calculations in the literature in which the standard oscillation phase is obtained assume the production of a coherent “neutrino flavour eigenstate” in the decay process, that is, a quantum superposition of mass eigenstates. This enforces equal production times for all mass eigenstates and hence the equal velocity condition. As pointed out in [3, 7, 8, 22] such a “neutrino flavour eigenstate” does not exist. The different neutrino mass eigenstates are produced incoherently in different physical processes. This universal, incorrect, assumption concerning the nature of the initial state of the neutrinos thus explains why the standard oscillation phase has been, hitherto, universally obtained. Thus the contribution to the oscillation phase of neutrino propagation has been generally underestimated by a factor of two, and the important contribution to the phase of the coherent source (resulting from different source decay times in the interfering amplitudes) has been universally neglected.

As discussed in some detail in Sect. 6, in order to enable the interference phenomenon, leading to “neutrino oscillations” to occur when the different mass eigenstates are produced at the same time inside the “neutrino flavour eigenstate” it was proposed [30] to introduce a spatial wave packet to delocalise the neutrinos. Moving with different velocities along classical trajectories, and produced at the same space-time point, it is clear the neutrinos can never arrive together at the unique space-time point of the detection event. The spatial “fuzziness” introduced by the hypothetical wave packets, associated with each neutrino, was conjectured to enable them both to have

non-vanishing amplitudes at the position of the detection event, so that neutrino oscillations can occur. In fact, because the interfering neutrinos can be produced at different times there is no need, to produce neutrino oscillations, for the spatial fuzziness introduced by the hypothetical wave packets.

The hypothetical wave packet suggested in [30] does not exist in the QM calculation of the neutrino production process. The neutrinos (particles) are produced in space-time according to an exponential decay law. A “wave packet” is only a (very loose) analogy in classical wave theory to the effect of the lifetime of an unstable particle on the energy (or mass) distribution of its decay products. In the analogy this distribution is given by the Fourier transform of the spatial wave packet, but there is no such wave packet in the QM calculation itself. Thus the (mathematically convenient) Gaussian wave packets that abound in the literature on the QM of neutrino oscillations have no physical foundation within QM. The relevant (related) parameter, in the quantum mechanical calculation, is the mean lifetime of the source. There is no physical connection between this parameter and the length of a loosely analogous wave packet with an arbitrary Gaussian form. The introduction of such wave packets in the QM calculation mixes up in a confused way concepts from QM and a classical wave theory that contains no information on the space-time evolution of particle positions.

The most important additional features of the present paper as compared to [3, 6] are, first, the realisation that the standard oscillation phase follows *only* from the manifestly unphysical equal velocity hypothesis and is quite unrelated to the use of Gaussian wave packets or any of the other kinematical assumptions made in the derivations. The misleading impression may have been given in [3, 6] that the standard oscillation phase, when obtained in covariant calculations, was a consequence of the use of wave packets. The second is the realisation that the incoherent nature of the neutrino production process, which is the physical basis of the calculations presented in [3, 6] and Sect. 5 above, was already pointed out more than twenty years ago by Shrock [7, 8] in the published literature. I was not aware of this work at the time of writing of [3, 6].

In closing a few remarks are made on phenomenology and experimental tests. The mass difference, $\Delta m_{\text{stand}}^2$, derived from experimental results using the standard formula, is related to that given by the Feynman path amplitude calculation, Δm_{FP}^2 , by the formula (valid for any source at rest)

$$\Delta m_{\text{FP}}^2 \equiv C \Delta m_{\text{stand}}^2 = \frac{\Delta m_{\text{stand}}^2}{\frac{E_S}{p_\nu} - 2}, \quad (99)$$

while for neutrino oscillations following two body decays in flight of ultra-relativistic charged pions or kaons [3]:

$$\Delta m_{\text{FP}}^2 = \frac{(1 - R_m^2) \Delta m_{\text{stand}}^2}{2R_m^2}, \quad (100)$$

where R_m is defined after (13). In the case of experiments involving neutrino production in pion, kaon or muon decay

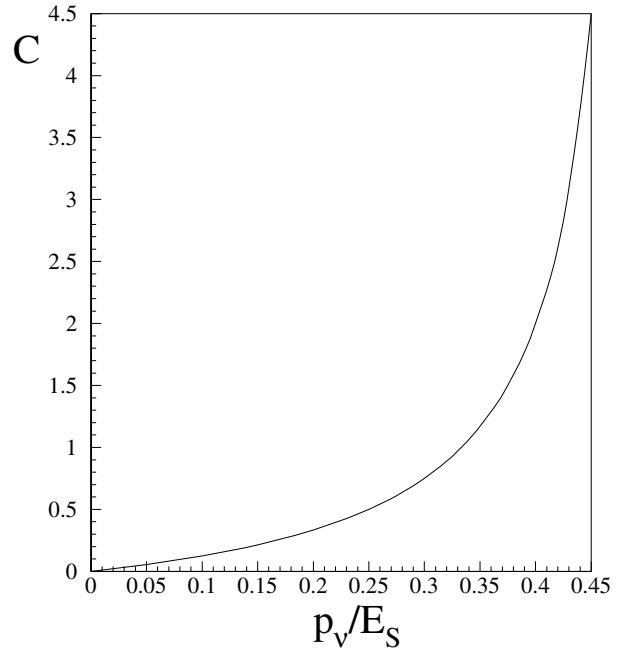


Fig. 1. Correction factor for Δm^2 relating the standard formula to the Feynman path amplitude calculation

the conversion is straightforward. However, for neutrinos produced in nuclear reactors the effective oscillation phase will require a suitable average, with appropriate weighting factors, over the decays of all β unstable nuclei contributing to the neutrino flux. For a given nuclear species the correction factor, C , to the standard oscillation phase is given, by (99), as $1/(E_\beta/p_\nu - 2)$, where E_β is the total energy release in the decay. This is evidently an immense undertaking for any actual reactor-based experiment. Any phenomenological conclusions hitherto drawn from the results of such experiments, using the standard formula, must therefore be discarded if the oscillation phase is correctly given by the path amplitude calculation.

Of particular interest, in view of the recent results of the Kamiokande [52] and SNO [53] collaborations, are the β decay processes ${}^8\text{B} \rightarrow {}^8\text{Be}^* + e^+ + (\nu_1, \nu_2, \nu_2)$ contributing to the flux of high energy solar neutrinos. The correction factor, C , in (99), is plotted in the range $0 < p_\nu/E_S < 0.45$ in Fig. 1. For ${}^8\text{B}$ β decay, $E_S \simeq 28$ MeV. The correction factor is unity when $p_\nu/E_S = 1/3$, corresponding to $p_\nu \simeq 9.3$ MeV. Near the kinematical end-point at $p_\nu \simeq 14$ MeV, C rises steeply, reaching a maximum value of about 1500. Thus neutrino oscillations, observable when $C \simeq 1$, are strongly suppressed¹² in the near end-point region. For the case of the electron capture reactions $e^- + {}^7\text{Be} \rightarrow {}^7\text{Li} + (\nu_1, \nu_2, \nu_2)$ contributing line spectra to the solar neutrino flux, $p_\nu \simeq E^*$, where E^* is the excitation energy of the unstable ${}^7\text{Be}$ atom and $C = -1$, so that the oscillation phase of the path amplitude calculation is the same as that given by the standard formula.

¹² i.e. they correspond to a vanishing mass difference or an infinite wavelength for the oscillation in the standard formula

As already mentioned in [3], evidence for neutrino oscillations in short baseline experiments such as LNSD [54] and KARMEN [55] can be confirmed or invalidated by a search for muon oscillations following pion decay at rest, since the muon oscillation phase is found [3] to be identical to that of the associated neutrinos given in (87) above. Evidently the event rate in such muon oscillation experiments can exceed that possible in the search for the associated neutrino oscillations by many orders of magnitude.

Note added in proof: It has demonstrated recently that coherent production of neutrino flavour eigenstates in pion decay is inconsistent with measurements of the ratio: $\Gamma(\pi \rightarrow e\nu)/\Gamma(\pi \rightarrow \mu\nu)$ and the MNS matrix elements [56]. In the same paper a definitive test of the oscillation phase formula, by comparing ' ν_μ disappearance' following either pion or kaon decay in long-baseline terrestrial experiments, is proposed.

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